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Topological Approximation Methods for Evolutionary Problems of Nonlinear Hydrodynamics

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Preface

The well-known methods of investigation for evolutionary problems of fluid dynamics are the Faedo–Galerkin method [36, 37, 61], the iteration method [2, 3], the method of evolution equations in Banach spaces [51, 78] and some others. In the present book we describe one more method of study of such problems, which is especially appropriate for the research of weak solvability for initial-boundary value problems arising in nonlinear hydrodynamics. Here we use this method for the investigation of some models for motion of viscoelastic media, its employment for other models can be found e.g. in works [29, 64, 71]. The outline of application of this method is the following one.

One begins with the interpretation of an initial-boundary value problem as an operator equation in the function space which naturally corresponds to the considered problem. As a rule, the maps involved in this equation do not possess good operator properties, so at once it is not possible to apply any principles of nonlinear analysis for the proof of the solvability of the problem. Therefore one finds an approximation of this equation (which consists in smoothing of nonlinear terms, or in adding terms of higher order with a small parameter, or in some other operations) and studies the solvability of this approximating equation in the spaces with more suitable topological properties. For this purpose, one applies the technique of the Leray–Schauder topological degree or its generalizations. It is important to point out that in this situation the approximating equation has natural equation properties, and small variations of the right-hand side and of the initial data imply small variation of the solution set for this equation. In particular, this gives opportunity to apply various approximate methods for the analysis of this equation, and the convergence of the approximate solutions to the solutions of the approximating equation is guaranteed. The last step of the method is the passage to the limit in the approximating equation as the approximation parameters tend to zero, and here the solutions of the approximating equation converge to a solution of the original equation (usually in a topology which is much weaker than the one of the spaces where the approximating equation was studied).

In particular, this method turned out to be useful in those problems of non-Newtonian hydrodynamics where it is hard or impossible to express the deviatoric stress tensor via the velocity vector function explicitly (and, consequently, one has to examine systems of two basic variables: the velocity vector and the stress tensor).

The book contains preliminary material from rheology, which is required for understanding the models under consideration. This material is written from the mathematician's point of view. Besides, for the sake of relative completeness of the research, we give some results on the existence of strong solutions for the initial-boundary value problems describing the motion of the media which are modeled by the considered models. Let us turn to more detailed description of the book's contents.

The **first** chapter of the book is the just mentioned introduction to the rheology of viscoelastic and nonlinear-viscous media. Let us point out that we do not try to describe all existing models, and we mainly restrict ourselves to the ones which are the objects of our research in the subsequent chapters. Section 1.1 is a summary of the basic laws which are independent of particular media. Section 1.2 is devoted to the brief analysis of one-dimensional viscoelastic models, which are generalized to higher dimensions in Section 1.3. The nonlinear-viscous models are handled in Section 1.4. In Section 1.5, we combine the viscoelastic and nonlinear-viscous approaches.

The **second** chapter is concerned with the function spaces which are required for the mathematical study of the hydrodynamical equations and with the interrelations between these spaces (embedding theorems etc.). Section 2.1 deals with the spaces of functions defined on subsets of $\mathbb{R}^n$ and with values in finite-dimensional spaces. In Section 2.2, we give the necessary results which involve the vector functions with values in Banach spaces.

The **third** chapter contains miscellaneous facts on linear (Section 3.1) and nonlinear (Section 3.2) non-evolutionary and evolutionary operator equations in Banach spaces, which we need for the investigation of equations of hydrodynamics (in particular, the notion of Leray–Schauder degree).

The **fourth** chapter is a rather general self-contained theory of attractors for evolutionary equations in Banach spaces. It is used in Chapter 6 for construction of attractors for the weak solutions to the initial-boundary value problems from the dynamics of viscoelastic media. Section 4.1 contains the classical issues of attractor theory; its generalizations for the evolution equations without uniqueness of solutions, without invariance of the trajectory space etc. are given in Section 4.2 (autonomous case) and Section 4.3 (non-autonomous case).

The goal of the **fifth** chapter is to present some results on the strong solvability and some solution properties for the initial-boundary value and Cauchy problems describing the motion of viscoelastic media. Unfortunately, as in the classical dynamics of Newtonian fluids in three-dimensional domains, for the problems under consideration one cannot prove global in time existence of strong solutions for arbitrary data even in two dimensions (although there is no counterexample). Section 5.1 is devoted to a local existence result for the initial-boundary value problem for the system of equations of motion for the Jeffreys viscoelastic medium. In the subsequent sections of the chapter we study the initial-value problem for the system of motion equations for a more general combined class of models for nonlinear viscoelastic media in the whole space $\mathbb{R}^2$ or $\mathbb{R}^3$. In Section 5.2, we formulate the problem and the main (existence and uniqueness) result and introduce additional notations. In Section 5.3, the operator treatment of the considered problem is realized. In Section 5.4, an auxiliary problem depending on a parameter is introduced and investigated. The existence of solutions of this problem and a uniform a priori estimate are proved. In Section 5.5, the passage to the limit as the parameter tends to zero is carried out, and the global strong solution of the original problem is obtained (for small initial data and body force). In Section 5.6, we study the continuous dependence of solutions on the problem data.

As we already mentioned, the problem of existence of global strong solutions to the initial-boundary value problems for the equations of viscoelastic fluid mechanics is open in the general case. A possible way to break this deadlock is to investigate weak solutions to these problems. This is realized in the **sixth** chapter. In Section 6.1, we give a set-theoretic scheme for weak formulation of problems and prove some important equalities. In Section 6.2, we introduce the concept of a weak solution to the initial-boundary value problem for the system of motion equations for the Jeffreys viscoelastic medium, and then we formulate the main existence theorem for it. In Section 6.3, existence of solutions of an auxiliary problem, depending on several parameters, is proved via obtaining a priori estimates and application of the Leray–Schauder degree theory. In Section 6.4, the passage to the limit as one of these parameters tends to zero is carried out. With the help of the obtained result, in Section 6.5 we prove existence of a weak solution of the initial-boundary value problem for the Jeffreys model (here we also touch on the existence of pressure).

The uniqueness of weak solutions for the majority of equations of hydrodynamics is still an open problem. For example, for the Navier–Stokes equations in the two-dimensional case a weak solution is unique, and in the three dimensions there are only conditional results. In Section 6.6 we present two results of the latter kind for the Jeffreys model. Section 6.7 (autonomous case) and Section 6.8 (non-autonomous case) are devoted to the study of attractors of weak solutions for the system of motion equations for the Jeffreys viscoelastic medium. In Section 6.9, we investigate stationary (independent of time) solutions of this system.

The **seventh** chapter represents another approach to the study of the equations of viscoelastic fluid mechanics. It is based on regularization ideas, and the problems arising here contain less unknown functions. In Section 7.1 we describe the regularization procedure for the equations of motion of the Jeffreys viscoelastic medium. We give the weak statement of the regularized initial-boundary value problem and formulate some existence results. In Sections 7.2 – 7.4, we apply the approximating-topological procedure for the proof of these theorems. In Section 7.5, we suggest another weak formulation of the regularized problem and compare it with the first one. This helps to establish (in Section 7.6) the convergence of the solutions of the regularized problems to the solutions of the original problem in some generalized sense. Section 7.7 is a sort of appendix to Chapter 7. Here we describe necessary constructions of regularization operators.

Voronezh, November 2007

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Contents

Chapter 1

Non-Newtonian flows

1.1 Principles of flow description

1.1.1 The basic characteristics of a flow

For mathematical description of behaviour of the real fluids and of the media close to fluids, in hydrodynamics it is usually supposed that the particles of a medium are infinitesimal, and they are situated at the points of an open set (domain) of the n-dimensional space. The domain corresponds to a vessel filled with this medium. In the course of time, the particles move and describe some trajectories in the space. The "vessel" can also move with the course of time; it can have elastic boundary, which changes its shape under the influence of the flow; it can have holes, through which the medium flows in or out, and so on.

Remark 1.1.1. The most typical situation is $n = 3$; the case $n = 2$ corresponds to the so called plane-parallel flows, but from the cognitive point of view one can also investigate other cases.

The problem of description of motion for a medium may be reduced to the description of motion for each point particle of the medium. Consider such a particle. Its position is described by a function $x(t)$ of time t with values in the points of the space.

Assume that in the space an origin and an orthonormal basis are fixed. In this case the coordinates of a point (or of a vector) are denoted as $x = (x_1, x_2, \dots, x_n)$. Thus the space is identified with the arithmetical n-dimensional space $\mathbb{R}^n$.

The velocity of the particle with a trajectory $x(t)$ at the moment t is

$$v(t) = x'(t). \tag{1.1.1}$$

Remark 1.1.2. Here and below in this chapter we consider all functions to be sufficiently smooth, so that all required derivatives exist.

Definition 1.1.1. A linear operator transforming (n-dimensional) vectors to vectors will be briefly called *tensor*.

Remark 1.1.3. Provided a basis is chosen, a tensor may be identified with a matrix $n \times n$. The elements of this matrix will be called components of the tensor.

Remark 1.1.4. The tensor is called orthogonal, symmetric etc., if the corresponding operator (or matrix) is orthogonal, symmetric and so on.

Denote by $v(t,x)$ the velocity of the particle which at the moment of time t is situated at the spatial point x. Consider the gradient of this function, i.e. the tensor with the components

$$(\nabla v)_{ij}(t,x) = \frac{\partial v_i(t,x)}{\partial x_j}. \tag{1.1.2}$$

Its symmetrical part

$$\mathcal{E} = \frac{1}{2}(\nabla v + \nabla v^\top) \tag{1.1.3}$$

is called the *strain velocity tensor*, and the skew-symmetrical one

$$W = \frac{1}{2}(\nabla v - \nabla v^\top) \tag{1.1.4}$$

is called the *vorticity tensor*.

Consider a particle inside the medium. Assume that it is surrounded by an imaginary surface of an arbitrary shape. The remaining part of the medium effects the particle through this surface. Consider a small flat part of this surface of area ΔS with the exterior normal vector $\overrightarrow{n}$. Denote by $P_{\overrightarrow{n}}$ the force of action of the outer part of the medium onto the particle through the considered area ΔS.

The vector

$$p_{\overrightarrow{n}} = \lim_{\Delta S \to 0} \frac{P_{\overrightarrow{n}}}{\Delta S}$$

is called *stress*.

The fundamental Cauchy theorem [63] says that there exists a symmetric tensor $\mathrm{T}(t,x)$ such that the stress at the moment t at the point x in the direction $\overrightarrow{n}$ is expressed by the formula:

$$p_{\overrightarrow{n}}(t,x) = \mathrm{T}(t,x)\overrightarrow{n}.$$

This tensor is called the *stress tensor*.

The tensor

$$\sigma = \mathrm{T} - \frac{1}{n}\operatorname{Tr}\mathrm{T}\, I, \tag{1.1.5}$$

where I is the unit tensor, is called the *deviatoric stress tensor*. Roughly speaking, it characterizes the forces of interior friction in a medium.

1.1.2 Newtonian fluid

The science that treats deformation and flow of materials is called *rheology*. The *rheological* behavior of a particular material depends on the relations between stresses, strains, stretchings in it. The most important of such relations is the *constitutive equation* (it is also called *constitutive law, constitutive relation*), which gives the connection between the deviatoric stress tensor σ and various characteristics of deformation.

The most simple relation of this kind, describing a fluid, looks like

$$\sigma = 2\eta\big(\mathcal{E} - \frac{1}{n}\operatorname{Tr}\mathcal{E}\, I\big), \tag{1.1.6}$$

where η is a scalar parameter called *viscosity*.

When $\eta = 0$, the fluid is called *ideal*. For $\eta > 0$ we have the classical *Newtonian fluid*. It is the basic object of the classical hydrodynamics. In this book, however, our prime interest is in the models different from the Newtonian one.

1.1.3 Equation of motion

Assume that the velocity field $v(t, x)$ is given at every geometrical point x of a spatial domain (where the medium is moving) and at every moment t in some time interval. Then, to describe the motion of the particle which at the initial moment t_0 is at a point x_0, it suffices to solve the Cauchy problem

$$x'(t) = v(t, x(t)), \tag{1.1.7}$$

$$x(t_0) = x_0. \tag{1.1.8}$$

If the velocity field v is regular enough, this problem has a unique solution.

Hence, the crucial problem for description of the medium motion is to find the velocity field $v(t, x)$. The basic tool to realize this is to use the following relation between the velocity and the stress tensor, which is called the *equation of motion* [10, 23]:

$$\rho\frac{\partial v}{\partial t} + \rho\sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \operatorname{Div}\mathrm{T} = \rho f. \tag{1.1.9}$$

Here $f(t, x)$ is the *body force* (i.e. the mass density of exterior forces effecting the particles of the material, for example, gravitational forces), $\rho(t, x)$ is the density of the medium, while the divergence of the tensor $\operatorname{Div}\mathrm{T}(t, x)$ is the vector

$$\left(\sum_{j=1}^{n}\frac{\partial \mathrm{T}_{1j}(t,x)}{\partial x_j}, \sum_{j=1}^{n}\frac{\partial \mathrm{T}_{2j}(t,x)}{\partial x_j}, \dots, \sum_{j=1}^{n}\frac{\partial \mathrm{T}_{nj}(t,x)}{\partial x_j}\right)$$
$$= \sum_{j=1}^{n}\left(\frac{\partial \mathrm{T}_{1j}(t,x)}{\partial x_j}, \frac{\partial \mathrm{T}_{2j}(t,x)}{\partial x_j}, \dots, \frac{\partial \mathrm{T}_{nj}(t,x)}{\partial x_j}\right).$$

For a *homogeneous incompressible* medium, one has more information. Firstly, the divergence of velocity is zero:

$$\operatorname{div} v(t,x) = \sum_{j=1}^{n} \frac{\partial v_j(t,x)}{\partial x_j} = 0. \tag{1.1.10}$$

Secondly, because of the constraints generated by the incompressibility, the trace of the stress tensor has to be considered as completely independent of the deformation characteristics, so one has to introduce the unknown scalar function $p(t,x) = -\frac{1}{n}\operatorname{Tr}\mathrm{T}$, called the *hydrostatic pressure* (see [63], Chapter IV, §7). Formula (1.1.5) implies

$$\mathrm{T} = -pI + \sigma. \tag{1.1.11}$$

Thirdly, the density is constant, and it is possible to consider it equal to one.

Note that $\operatorname{grad} p = (\frac{\partial p}{\partial x_1}, \dots, \frac{\partial p}{\partial x_n}) = \operatorname{Div}(pI)$. Therefore, (1.1.9) and (1.1.5) yield

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \operatorname{Div}\sigma + \operatorname{grad} p = f. \tag{1.1.12}$$

This is the equation of motion for homogeneous incompressible media.

We have now

$$\begin{aligned} 2\operatorname{Div}\mathcal{E} &= 2\sum_{j=1}^{n}\left(\frac{\partial \mathcal{E}_{1j}}{\partial x_j}, \frac{\partial \mathcal{E}_{2j}}{\partial x_j}, \dots, \frac{\partial \mathcal{E}_{nj}}{\partial x_j}\right) \\ &= \sum_{j=1}^{n}\left(\frac{\partial^2 v_1}{\partial x_j^2}, \frac{\partial^2 v_2}{\partial x_j^2}, \dots, \frac{\partial^2 v_n}{\partial x_j^2}\right) + \sum_{j=1}^{n}\left(\frac{\partial^2 v_j}{\partial x_j \partial x_1}, \frac{\partial^2 v_j}{\partial x_j \partial x_2}, \dots, \frac{\partial^2 v_j}{\partial x_j \partial x_n}\right). \end{aligned}$$

Condition (1.1.10) implies that the second sum is equal to zero. Therefore

$$2\operatorname{Div}\mathcal{E} = \Delta v, \tag{1.1.13}$$

where Δ is the Laplacian $\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$.

Then from Newton's constitutive relation (1.1.6) and equation of motion (1.1.12) we get the equation of motion for the Newtonian fluid:

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \eta\Delta v + \operatorname{grad} p = f, \tag{1.1.14}$$

which is usually called the Navier–Stokes equation.

1.1.4 No-slip condition

Assume that the medium is contained in a motionless vessel which may be identified with an open set (domain) Ω in the n-dimensional space $\mathbb{R}^n$ (see Remark 1.1.1). The *no-slip condition* means that the velocity of the medium vanishes on the boundary of Ω:

$$v(t,x) = 0, \ x \in \partial\Omega. \tag{1.1.15}$$

A justification for the no-slip condition may be found if we remember that a real medium consists of molecules. When a medium is stationary, its molecules still move with a random velocity, but the mean velocity of this motion is zero. When the medium begins to move, there appears a non-zero mean velocity, sometimes called the bulk velocity, which is superimposed on the random velocity of the stationary state. Just this bulk velocity corresponds to the velocity (1.1.1) at the idealized hydrodynamical approach described in Section 1.1.1. Near the interface between the medium and the solid boundary of the vessel there exists an attraction between the molecules of the medium and the molecules of the solid boundary. The attractive force is usually so strong that the bulk velocity of the medium vanishes at this interface.

The no-slip condition can be illustrated by the following folklore example. If you like Italian or French cuisine, your dishes may be sometimes stained with cheese. When you put them into your dishwasher, you realize that they are not washed up well. This is not surprising, since, due to the no-slip condition, the velocity vanishes at the interface between the water and the cheese, so the stream itself is not able to carry away the cheese. A possible way out is to scrub the cheese by yourself.

1.2 One-dimensional models of viscoelastic media

1.2.1 Method of mechanical models

In rheology, for construction of constitutive relations describing materials with more complicated properties than the Newtonian fluid, the method of mechanical models is often used. Let us describe the essence of this heuristic method.

The basic properties for rheology are *elasticity, viscosity* and *plasticity*. For representation of the elasticity a spiral *spring* is used. It satisfies Hooke's law: the length variation of the spring is in direct proportion to the force applied to its ends. This model is denoted by the letter H. For representation of viscosity one uses a viscous damper (*dashpot*). The velocity of the dashpot (i.e. the rate of change of its length) is in direct proportion to the applied force. This model is denoted by N. The model for plasticity is not used in this book.

Assume that investigated media consist of microcomplexes of small springs and dashpots. Within these complexes they are connected in parallel (denoted by $|$) or in series (denoted by $-$). For the parallel connection the loadings perceived by each element are summarized and the velocities of each element are the same. For the connection in series the velocities are summarized and the loadings are the same.

However for rheological relations one does not need the forces and the velocities but the stresses and strain velocities. The stress (in a spring or dashpot) may be understood as the ratio of the resistance force (which is equal in absolute value to the applied force) to the cross-section area (of the spring or dashpot). The strain velocity ε may be understood as a half of the ratio of the velocity of the spring or dashpot (i.e. the rate of change of the length) to the average longitudinal length of the spring or dashpot.

Now one has to write relations between the stress and the strain velocity for the spring and the dashpot. For the dashpot it is very simple. If the force is in direct proportion to the velocity, then the stress is in direct proportion to the strain velocity:

$$\sigma_N = 2\eta\varepsilon_N. \tag{1.2.1}$$

Remark 1.2.1. Note that (1.2.1) resembles (1.1.6), so η has the physical sense of viscosity.

Now, differentiating (with respect to time) Hooke's law for the spring, we conclude that the derivative of the force is in direct proportion to the velocity of variation of length of the spring. Therefore the derivative of the stress is in direct proportion to the strain velocity:

$$\dot{\sigma}_H = 2\mu\varepsilon_H. \tag{1.2.2}$$

From here (but only in Section 1.2) the time derivative is denoted by a point. The constants of proportionality μ and η are usually positive.

1.2.2 The Maxwell body

The simplest structure of a model for a viscoelastic medium is the Maxwell body with the symbolical notation $M = H - N$, i.e. a spring and a dashpot connected in series. Let us deduce the constitutive equation for this body. For this purpose, recall that for the connection in series the stress is constant

$$\sigma_M = \sigma_N = \sigma_H, \tag{1.2.3}$$

and the strain velocities are summarized:

$$\varepsilon_M = \varepsilon_H + \varepsilon_N. \tag{1.2.4}$$

From equalities (1.2.1) – (1.2.4) it follows that

$$\varepsilon_M = \frac{\dot{\sigma_M}}{2\mu} + \frac{\sigma_M}{2\eta}. \tag{1.2.5}$$

This is the constitutive relation for the Maxwell body. It looks like a linear differential equation for σ_M. Multiplying both parts of this equation by $e^{\frac{\mu}{\eta}t}$, we get:

$$e^{\frac{\mu}{\eta}t}\mathcal{E}_M = e^{\frac{\mu}{\eta}t}\frac{\dot{\sigma_M}}{2\mu} + e^{\frac{\mu}{\eta}t}\frac{\sigma_M}{2\eta},$$

$$e^{\frac{\mu}{\eta}t}\mathcal{E}_M = \frac{(e^{\frac{\mu}{\eta}t}\sigma_M)^{\boldsymbol{\cdot}}}{2\mu}.$$

It implies

$$e^{\frac{\mu}{\eta}t}\sigma_M = \sigma_{M_0} + 2\mu\int_0^t e^{\frac{\mu}{\eta}s}\mathcal{E}_M(s)\,ds,$$

σ_{M_0} is the stress at the initial moment (here and below it is the moment $t = 0$). We conclude that the solution of equation (1.2.5) has the following form:

$$\sigma_M = e^{-\frac{\mu}{\eta}t}\Big(\sigma_{M_0} + 2\mu\int_0^t e^{\frac{\mu}{\eta}s}\mathcal{E}_M(s)\,ds\Big). \tag{1.2.6}$$

1.2.3 The Jeffreys body

The most typical model of a viscoelastic medium is the Jeffreys model J. It consists of a Maxwell body and one more dashpot, connected in parallel, i.e. $J = M|N$. The viscosities of mediums in two different dashpots may differ; denote them by η_1 and η_2. Thus, for the Maxwell constituent of the Jeffreys body we have:

$$\sigma_M = e^{-\frac{\mu}{\eta_1}t}\Big(\sigma_{M_0} + 2\mu\int_0^t e^{\frac{\mu}{\eta_1}s}\mathcal{E}_M(s)\,ds\Big), \tag{1.2.7}$$

and for the Newtonian constituent

$$\sigma_N = 2\eta_2\mathcal{E}_N. \tag{1.2.8}$$

But for the parallel connection one has:

$$\sigma_J = \sigma_M + \sigma_N, \tag{1.2.9}$$

$$\mathcal{E}_J = \mathcal{E}_M = \mathcal{E}_N. \tag{1.2.10}$$

Equalities (1.2.7) – (1.2.10) yield

$$\sigma_J = e^{-\frac{\mu}{\eta_1}t}\Big(\sigma_{M_0} + 2\mu\int_0^t e^{\frac{\mu}{\eta_1}s}\mathcal{E}_J(s)\,ds\Big) + 2\eta_2\mathcal{E}_J. \tag{1.2.11}$$

Let us denote the stress and the strain velocity of the Jeffreys body at the initial moment of time by σ_{J_0} and $\mathcal{E}_{J_0}$. Taking $t = 0$ in (1.2.11), we get the equality:

$$\sigma_{J_0} = \sigma_{M_0} + 2\eta_2\mathcal{E}_{J_0}. \tag{1.2.12}$$

Then (1.2.11) and (1.2.12) imply

$$\sigma_J = e^{-\frac{\mu}{\eta_1}t}\left(\sigma_{J_0} - 2\eta_2 \varepsilon_{J_0} + 2\mu \int_0^t \varepsilon_J(s) e^{\frac{\mu}{\eta_1}s}\, ds\right) + 2\eta_2 \varepsilon_J. \tag{1.2.13}$$

This is the rheological relation for the Jeffreys body with explicitly expressed stress. Another form of the constitutive law for the Jeffreys body, which we deduce just below, is more traditional.

Differentiate expression (1.2.11) with respect to t:

$$\dot{\sigma}_J = e^{-\frac{\mu}{\eta_1}t} 2\mu e^{\frac{\mu}{\eta_1}t} \varepsilon_J(t) - \frac{\mu}{\eta_1} e^{-\frac{\mu}{\eta_1}t}\left(\sigma_{M_0} + 2\mu \int_0^t e^{\frac{\mu}{\eta_1}s} \varepsilon_J(s)\, ds\right) + 2\eta_2 \dot{\varepsilon}_J. \tag{1.2.14}$$

On the other hand, (1.2.11) implies:

$$e^{-\frac{\mu}{\eta_1}t}\left(\sigma_{M_0} + 2\mu \int_0^t e^{\frac{\mu}{\eta_1}s} \varepsilon_J(s)\, ds\right) = \sigma_J - 2\eta_2 \varepsilon_J. \tag{1.2.15}$$

Equalities (1.2.14) and (1.2.15) yield

$$\dot{\sigma}_J = 2\mu \varepsilon_J - \frac{\mu}{\eta_1}(\sigma_J - 2\eta_2 \varepsilon_J) + 2\eta_2 \dot{\varepsilon}_J.$$

Multiply this by $\frac{\eta_1}{\mu}$ and move the term σ_J to the left-hand side:

$$\sigma_J + \frac{\eta_1}{\mu}\dot{\sigma}_J = 2(\eta_1 + \eta_2)\varepsilon_J + 2\frac{\eta_1 \eta_2}{\mu}\dot{\varepsilon}_J. \tag{1.2.16}$$

Denote $\lambda_1 = \frac{\eta_1}{\mu}$, $\lambda_2 = \frac{\eta_1 \eta_2}{\mu(\eta_1+\eta_2)}$, $\eta_J = \eta_1 + \eta_2$. Then (1.2.16) may be rewritten as

$$\sigma_J + \lambda_1 \dot{\sigma}_J = 2\eta_J(\varepsilon_J + \lambda_2 \dot{\varepsilon}_J). \tag{1.2.17}$$

That is how another form of the constitutive equation for the Jeffreys body looks like. The parameter η_J is called the viscosity of the Jeffreys body. The parameter λ_1 is called the *relaxation time*, and the parameter λ_2 is the *retardation time*. Since the viscosities are positive and $\lambda_2 = \lambda_1 \frac{\eta_2}{\eta_1+\eta_2}$, one has

$$\lambda_2 < \lambda_1.$$

From the physical point of view, the parameters $\lambda_1, \lambda_2, \eta_J$ are preferable with respect to the parameters μ_1, η_1, η_2 because the first ones are not only parameters of a mechanical model, but can be measured for real media possessing viscoelastic properties of the Jeffreys model.

Assume that the strain velocity is constant and equal to ε_{J_0}. Then $\dot{\varepsilon}_J = 0$, and equation (1.2.17) may be transformed as follows:

$$\sigma_J - 2\eta_J \varepsilon_{J_0} + \lambda_1 (\sigma_J - 2\eta_J \varepsilon_{J_0})' = 0.$$

Solving this differential equation, we obtain

$$\sigma_J - 2\eta_J \varepsilon_{J_0} = (\sigma_{J_0} - 2\eta_J \varepsilon_{J_0}) e^{-\frac{t}{\lambda_1}}.$$

We have the following expression for the stress:

$$\sigma_J = 2\eta_J \varepsilon_{J_0} + (\sigma_{J_0} - 2\eta_J \varepsilon_{J_0}) e^{-\frac{t}{\lambda_1}}. \tag{1.2.18}$$

If $\sigma_{J_0} = 2\eta_J \varepsilon_{J_0}$, then $\sigma_J = 2\eta_J \varepsilon_{J_0}$, and the stress does not depend on time. If deformation in a body remains constant, the strain velocity ε_{J_0} is equal to zero, and the stress decreases according to the exponential law:

$$\sigma_J = \sigma_{J_0} e^{-\frac{t}{\lambda_1}}.$$

Thus, the relaxation time λ_1 is the time, during which, for constant deformation, the stress is reduced to $1/e$-th part.

If there is no stress ($\sigma_J = 0$), then (1.2.17) implies

$$\varepsilon_J + \lambda_2 \dot{\varepsilon}_J = 0,$$

and

$$\varepsilon_J = e^{-\frac{t}{\lambda_2}} \varepsilon_{J_0}, \tag{1.2.19}$$

i.e. the strain velocity decreases according to the exponential law, and during the retardation time λ_2 the strain velocity is reduced to $1/e$-th part.

Let us express the "mechanical" parameters as functions of the viscosity of the Jeffreys body and the times of relaxation and retardation:

$$\eta_2 = \eta_J \frac{\lambda_2}{\lambda_1}, \quad \eta_1 = \eta_J \big(1 - \frac{\lambda_2}{\lambda_1}\big), \quad \mu = \eta_J \frac{\lambda_1 - \lambda_2}{\lambda_1^2}.$$

Thus, (1.2.13) turns into

$$\sigma_J = e^{-\frac{t}{\lambda_1}} \big(\sigma_{J_0} - 2\eta_J \frac{\lambda_2}{\lambda_1} \varepsilon_{J_0} + 2\eta_J \frac{\lambda_1 - \lambda_2}{\lambda_1^2} \int_0^t e^{\frac{s}{\lambda_1}} \varepsilon_J(s)\, ds\big) + 2\eta_J \frac{\lambda_2}{\lambda_1} \varepsilon_J. \tag{1.2.20}$$

This is one more form of the constitutive relation for the Jeffreys body.

Remark 1.2.2. The equations (1.2.17) and (1.2.20) are also rheological relations for another viscoelastic body. It is the Lethersich body [49] with symbolical notation $(N|H) - N$, where the viscosities of two dashpots N may differ.

Remark 1.2.3. For $\lambda_2 = \eta_2 = 0$ the Jeffreys body turns to be equivalent to the Maxwell one: it suffices to compare (1.2.11) and (1.2.6).

1.3 Multidimensional models of viscoelastic media

1.3.1 Passage to multidimensional models

The employment of the models obtained in the previous section for description of mediums in the ”real” n-dimensional space (see Remark 1.1.1) requires generalization of the equations (first of all, of the constitutive laws) to the n-dimensional case. It is the last step of the method of mechanical models.

The problem is how to understand the components of the obtained constitutive relations, for example, of (1.2.17), in the n-dimensional case. Here the answer is that the times of relaxation and retardation of the Jeffreys body remain scalars, as well as the viscosity, so it is possible to measure them for particular materials. The stress and the strain velocity become the deviatoric stress tensor and the strain velocity tensor, respectively (see however Remarks 1.3.1 and 1.3.5). The only problem is how to understand the time derivative denoted by a point. There is no direct answer to this question, there are several variants (almost all of them, however, may be criticized), and the choice of a variant determines the obtained model. The following subsections are devoted to the analysis of the principal variants.

Remark 1.3.1. We will describe all the variants by the example of the Jeffreys model. Henceforth, the indices $_J$ at the symbols of viscosity, stress tensor and strain velocity tensor will be omitted for brevity. In Section 1.3 it is implicitely assumed that we deal with *incompressible medium*, i.e.

$$\operatorname{div} v = \operatorname{Tr} \mathcal{E} = 0.$$

However, the results remain valid for *compressible medium* with or without minor changes (such as to replace $\mathcal{E}$ with $\widetilde{\mathcal{E}} = \mathcal{E} - \frac{1}{n} \operatorname{Tr} \mathcal{E}\, I$ or to stop treating σ as a deviatoric stress tensor, cf. Remark 1.3.5 below).

1.3.2 Partial derivative

The simplest way out, which causes minor mathematical difficulties, is the choice of the partial derivative with respect to time t as a substitute for the derivative denoted by a point.

The constitutive relation (1.2.17) becomes

$$\sigma + \lambda_1 \frac{\partial \sigma}{\partial t} = 2\eta\big(\mathcal{E} + \lambda_2 \frac{\partial \mathcal{E}}{\partial t}\big), \tag{1.3.1}$$

and the equivalent constitutive relation (1.2.20) becomes

$$\sigma = e^{-\frac{t}{\lambda_1}} \left(\sigma_0 - 2\eta \frac{\lambda_2}{\lambda_1} \mathcal{E}_0 + 2\eta \frac{\lambda_1 - \lambda_2}{\lambda_1^2} \int_0^t e^{\frac{s}{\lambda_1}} \mathcal{E}(s)\, ds \right) + 2\eta \frac{\lambda_2}{\lambda_1} \mathcal{E}. \tag{1.3.2}$$

In the case of a homogeneous incompressible medium this and (1.1.13) imply:

$$\operatorname{Div}\sigma = e^{-\frac{t}{\lambda_1}}\left(\operatorname{Div}\sigma_0 - \eta\frac{\lambda_2}{\lambda_1}\Delta v|_{t=0} + \eta\frac{\lambda_1-\lambda_2}{\lambda_1^2}\int_0^t e^{\frac{s}{\lambda_1}}\Delta v(s)\,ds\right) + \eta\frac{\lambda_2}{\lambda_1}\Delta v.$$

Substituting it into the equation of motion (1.1.12), we get the following equation of motion for a homogeneous incompressible viscoelastic medium

$$\begin{aligned}\frac{\partial v}{\partial t} + \sum_{i=1}^n v_i\frac{\partial v}{\partial x_i} - \eta\frac{\lambda_2}{\lambda_1}\Delta v - \eta\frac{\lambda_1-\lambda_2}{\lambda_1^2}\int_0^t e^{\frac{s-t}{\lambda_1}}\Delta v(s)\,ds + \operatorname{grad} p \\ = f + e^{-\frac{t}{\lambda_1}}\big(\operatorname{Div}\sigma_0 - \eta\frac{\lambda_2}{\lambda_1}\Delta v|_{t=0}\big).\end{aligned} \tag{1.3.3}$$

1.3.3 Substantial derivative

Equation (1.3.1) describes the relation between stresses, strains and their partial derivatives with respect to time, i.e. with the rates of their change at each fixed geometrical point of space. It is more natural from different points of view to use the approach when the constitutive relation connects stresses and strains with the rates of their change for each particle. For this purpose, introduce an auxiliary function z. Let t and τ be two arbitrary moments of time and let x be a spatial point. Consider the particle which at the moment τ is at the spatial point x. Then $z(t,\tau,x)$ expresses the spatial position of this particle at the moment t.

From this definition we can see that

$$\frac{\partial z(t,\tau,x)}{\partial t} = v(t,z(t,\tau,x)), \tag{1.3.4}$$

$$z(t,t,x) = x, \tag{1.3.5}$$

$$z(s,t,z(t,\tau,x)) = z(s,\tau,x). \tag{1.3.6}$$

Let $A(t,x)$ be a sufficiently smooth function of time and space. The rate of change of A for a fixed particle is equal to

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{A(t+\Delta t, z(t+\Delta t,t,x)) - A(t,x)}{\Delta t} &= \frac{\partial}{\partial\tau}A(\tau,z(\tau,t,x))\Big|_{\tau=t} \\ = \frac{\partial A(\tau,x)}{\partial\tau}\Big|_{\tau=t} + \sum_{i=1}^n \frac{\partial A(\tau,x)}{\partial x_i}\frac{\partial z_i(\tau,t,x)}{\partial\tau}\Big|_{\tau=t} \\ = \frac{\partial A(t,x)}{\partial t} + \sum_{i=1}^n v_i(t,x)\frac{\partial A(t,x)}{\partial x_i}.\end{aligned} \tag{1.3.7}$$

This function is called the substantial derivative of A and is denoted by $\frac{dA}{dt}$. Observe that

$$\frac{dx}{dt} = \frac{\partial x}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial x}{\partial x_i} = \sum_{i=1}^{n} v_i e_i = v,$$

where e_i, $i = 1, 2, \ldots, n$, are the basis vectors of the space. Thus, we have the following property of the substantial derivative:

$$v(t, x) = \frac{dx}{dt}.$$

To obtain an analogue of (1.2.20), we have to use the inverse operation for the new derivation $\frac{d}{dt}$. This inverse operation is the integration along the trajectories of particles. Let us study this problem more formally.

We have to find the solution to the following Cauchy problem

$$\begin{aligned} \frac{d}{dt} U(t, x) &= A(t, x), \\ U(0, x) &= U_0(x), \end{aligned} \tag{1.3.8}$$

where A and U_0 are arbitrary sufficiently smooth known functions.

Let us show that

$$\frac{d}{dt} \int_0^t A(s, z(s, t, x))\, ds = A(t, x). \tag{1.3.9}$$

Really, the continued equality (1.3.7) gives

$$\frac{dA}{dt}(t, x) = \frac{\partial}{\partial \tau} A(\tau, z(\tau, t, x))\Big|_{\tau = t}. \tag{1.3.10}$$

Taking into account (1.3.6) and (1.3.10), we obtain:

$$\begin{aligned} \frac{d}{dt} \int_0^t A(s, z(s, t, x))\, ds &= \frac{\partial}{\partial \tau} \int_0^\tau A(s, z(s, \tau, z(\tau, t, x)))\, ds\Big|_{\tau = t} \\ &= \frac{\partial}{\partial \tau} \int_0^\tau A(s, z(s, t, x))\, ds\Big|_{\tau = t} = A(\tau, z(\tau, t, x))\Big|_{\tau = t} = A(t, x). \end{aligned}$$

Observe that the function $U_0(z(0, t, x))$ possesses the value $U_0(x)$ at $t = 0$. Besides, using (1.3.6) and (1.3.10), we obtain

$$\frac{d}{dt} U_0(z(0, t, x)) = \frac{\partial}{\partial \tau} U_0(z(0, \tau, z(\tau, t, x)))\Big|_{\tau = t} = \frac{\partial}{\partial \tau} U_0(z(0, t, x))\Big|_{\tau = t} = 0.$$

Thus, the solution of problem (1.3.8) has the following representation:

$$U_0(z(0, t, x)) + \int_0^t A(s, z(s, t, x))\, ds. \tag{1.3.11}$$

But this problem cannot have more than one solution. Really, let U_1 and U_2 be two solutions of problem (1.3.8), and $w = U_1 - U_2$. Then we have:

$$\frac{d}{dt}w(t,x) = 0, \quad w(0,x) = 0.$$

Equality (1.3.10) yields

$$\frac{\partial}{\partial \tau}w(\tau, z(\tau,t,x))\Big|_{\tau=t} = 0.$$

Substituting here $z(t,0,y)$ for x, where y is an arbitrary point, we get:

$$0 = \frac{\partial}{\partial \tau}w(\tau, z(\tau,t,z(t,0,y)))\Big|_{\tau=t} = \frac{\partial}{\partial \tau}w(\tau, z(\tau,0,y))\Big|_{\tau=t} = \frac{\partial}{\partial t}w(t, z(t,0,y)).$$

Let $w_1(t,y) = w(t, z(t,0,y))$. Then

$$\frac{\partial}{\partial t}w_1(t,y) = 0, \quad w_1(0,y) = 0.$$

Hence, $w_1 \equiv 0$. Then

$$0 = w_1(t, z(0,t,x)) = w(t, z(t,0,z(0,t,x))) = w(t,x)$$

for any x and t. Thus, $w \equiv 0$, and uniqueness of the solution for problem (1.3.8) is proven.

Now we can return to deriving of constitutive relations. Using the substantial derivative, rewrite the constitutive equation (1.2.17) as

$$\sigma + \lambda_1 \frac{d}{dt}\sigma = 2\eta(\mathcal{E} + \lambda_2 \frac{d}{dt}\mathcal{E}). \tag{1.3.12}$$

We are going to derive an explicit expression for σ, i.e. an analogue of (1.2.20). Let us carry out a number of transformations of equation (1.3.12):

$$\sigma - 2\eta\frac{\lambda_2}{\lambda_1}\mathcal{E} + \lambda_1 \frac{d}{dt}\Big(\sigma - 2\eta\frac{\lambda_2}{\lambda_1}\mathcal{E}\Big) = 2\eta\Big(\frac{\lambda_1-\lambda_2}{\lambda_1}\mathcal{E}\Big),$$

$$\frac{e^{\frac{t}{\lambda_1}}}{\lambda_1}\Big(\sigma - 2\eta\frac{\lambda_2}{\lambda_1}\mathcal{E}\Big) + e^{\frac{t}{\lambda_1}}\frac{d}{dt}\Big(\sigma - 2\eta\frac{\lambda_2}{\lambda_1}\mathcal{E}\Big) = 2\eta e^{\frac{t}{\lambda_1}}\Big(\frac{\lambda_1-\lambda_2}{\lambda_1^2}\mathcal{E}\Big),$$

$$\frac{d}{dt}\Big(e^{\frac{t}{\lambda_1}}\Big(\sigma - 2\eta\frac{\lambda_2}{\lambda_1}\mathcal{E}\Big)\Big) = 2\eta e^{\frac{t}{\lambda_1}}\Big(\frac{\lambda_1-\lambda_2}{\lambda_1^2}\mathcal{E}\Big).$$

In accordance with (1.3.11), the solution for this equation has the following form:

$$e^{\frac{t}{\lambda_1}}\Big(\sigma - 2\eta\frac{\lambda_2}{\lambda_1}\mathcal{E}\Big) = \sigma_0(z(0,t,x)) - 2\eta\frac{\lambda_2}{\lambda_1}\mathcal{E}_0(z(0,t,x)) + 2\eta\frac{\lambda_1-\lambda_2}{\lambda_1^2}\int_0^t e^{\frac{s}{\lambda_1}}\mathcal{E}(s, z(s,t,x))\,ds.$$

It implies an expression for σ:

$$\begin{aligned}\sigma(t,x) = e^{-\frac{t}{\lambda_1}}\big[\sigma_0(z(0,t,x)) - 2\eta\frac{\lambda_2}{\lambda_1}\mathcal{E}_0(z(0,t,x))\\ + 2\eta\frac{\lambda_1-\lambda_2}{\lambda_1^2}\int_0^t e^{\frac{s}{\lambda_1}}\mathcal{E}(s,z(s,t,x))\,ds\big] + 2\eta\frac{\lambda_2}{\lambda_1}\mathcal{E}(t,x).\end{aligned} \tag{1.3.13}$$

In the case of a homogeneous incompressible medium, (1.3.13) and (1.1.13) yield

$$\begin{aligned}\operatorname{Div}\sigma(t,x) = e^{-\frac{t}{\lambda_1}}\operatorname{Div}\big[\sigma_0(z(0,t,x)) - 2\eta\frac{\lambda_2}{\lambda_1}\mathcal{E}_0(z(0,t,x))\big]\\ + 2\eta\frac{\lambda_1-\lambda_2}{\lambda_1^2}\int_0^t e^{\frac{s-t}{\lambda_1}}\operatorname{Div}[\mathcal{E}(s,z(s,t,x))]\,ds + \eta\frac{\lambda_2}{\lambda_1}\Delta v.\end{aligned}$$

To get the equation of motion of the homogeneous incompressible Jeffreys' viscoelastic medium, substitute it into (1.1.12):

$$\begin{aligned}\frac{\partial v}{\partial t} + \sum_{i=1}^n v_i\frac{\partial v}{\partial x_i} - \eta\frac{\lambda_2}{\lambda_1}\Delta v - 2\eta\frac{\lambda_1-\lambda_2}{\lambda_1^2}\int_0^t e^{\frac{s-t}{\lambda_1}}\operatorname{Div}[\mathcal{E}(s,z(s,t,x))]\,ds + \operatorname{grad} p\\ = f + e^{-\frac{t}{\lambda_1}}\operatorname{Div}\big[\sigma_0(z(0,t,x)) - 2\eta\frac{\lambda_2}{\lambda_1}\mathcal{E}_0(z(0,t,x))\big].\end{aligned} \tag{1.3.14}$$

1.3.4 Principle of material frame-indifference. Frame-indifferent functions

Before passing to the description of other objects able to replace in the n-dimensional situation the time derivative denoted by a point, it is necessary to say a few words about the principle of material frame-indifference. It is one of the main principles of rational mechanics, which expresses the fact that the properties of a material do not depend on the choice of observer.

The observer in rational mechanics is identified with a frame, i.e. a certain correspondence between the spatial points and the elements x of the space $\mathbb{R}^n$, as well as between the moments of time and the elements t of the scalar axis $\mathbb{R}$. It is considered that when the observer is changed, the metrics in $\mathbb{R}^n$ and in the scalar axis are conserved, the time direction is also conserved. Then the most general change of coordinates for each point looks like [63]:

$$t^* = t + a, \tag{1.3.15}$$

$$x^* = x_0^*(t) + Q(t)(x - x_0), \tag{1.3.16}$$

where a is a time value, x_0 is a spatial point, $x_0^*(t)$ is a function of time with the values at the spatial points, Q is a time dependent orthogonal tensor.

A change of observer induces some transformation of vectors and tensors. However, the *principle of material frame-indifference* [63] states that such a transformation (i.e. a change of coordinates (1.3.15) – (1.3.16)) should not affect the structure of the formulas expressing physical properties of a medium and containing the time variable t, the spatial variable x and their functions.

Remark 1.3.2. In spite of these statements, it should be pointed out that the form of the equations of motion (1.1.9) and (1.1.12) depends on the choice of observer [63]. The point here is that, generally speaking, the equation of motion has form (1.1.9) only in *inertial* [63] reference frame. If a change of variables of the form (1.3.15) – (1.3.16) transforms inertial frames to inertial ones (it means, in particular, that Q is independent of time t), the equation of motion does not change its form.

Take an arbitrary vector, which is *geometrical*, i.e. it is a directed segment existing in the space irrespectively of the observer. Assume that in the initial frame of reference it has the form $w = \overrightarrow{x_1 x_2}$. Then in the new frame of reference $w^* = \overrightarrow{x_1^* x_2^*} = \overrightarrow{x_0^*(t) + Q(t)(x_1 - x_0),\, x_0^*(t) + Q(t)(x_2 - x_0)} = Q(t)\overrightarrow{x_1 x_2} = Q(t)w$.

Let some tensor $\mathcal{T}$ transform geometrical vectors to geometrical ones. Then in the initial frame of reference we have:

$$w_1 = \mathcal{T} w_2,$$

where w_1 and w_2 are two trial geometrical vectors. Q is an orthogonal tensor, so $Q(t)^\top Q(t) = I$ and

$$w_1^* = Q(t)w_1 = Q(t)\mathcal{T} Q(t)^\top Q(t)w_2 = \mathcal{T}^* w_2^*,$$

where $\mathcal{T}^* = Q(t)\mathcal{T} Q(t)^\top$.

Based on this, a vector-valued function $w(t,x)$ of time and space is called *frame-indifferent* if under the change of frame (1.3.15) – (1.3.16) its coordinates are transformed as follows:

$$w^*(t^*, x^*) = Q(t)w(t,x); \tag{1.3.17}$$

a tensor-valued function $\mathcal{T}(t,x)$ is called *frame-indifferent* if under the similar change one has:

$$\mathcal{T}^*(t^*, x^*) = Q(t)\mathcal{T}(t,x)Q(t)^\top. \tag{1.3.18}$$

The stress tensor T is an example of a frame-indifferent function [63].

Moreover, a scalar function $A(t,x)$ is called *frame-indifferent* if under the same change of frame:

$$A^*(t^*, x^*) = A(t,x). \tag{1.3.19}$$

The density $\rho(t,x)$ is an example of such a function.

1.3.5 The Zaremba–Żórawski theorem

For checking frame-indifference of constitutive relations we are going to use the following statement on the transformation of the strain velocity and vorticity tensors under a change of observer; it is called the Zaremba–Żórawski theorem.

Theorem 1.3.1. *Under the frame change (1.3.15) – (1.3.16), the following transformations occur:*

$$\mathcal{E}^*(t^*, x^*) = Q(t)\mathcal{E}(t, x)Q(t)^\top, \tag{1.3.20}$$

i.e. the strain velocity tensor is frame-indifferent;

$$W^*(t^*, x^*) = Q(t)W(t, x)Q(t)^\top + Q'(t)Q(t)^\top, \tag{1.3.21}$$

i.e. the vorticity tensor is not frame-indifferent.

Proof. By definition, $z(t, \tau, x)$, for fixed arguments, is a point in the space. Therefore, (1.3.16) gives:

$$z^*(t^*, \tau^*, x^*) = x_0^*(t) + Q(t)(z(t, \tau, x) - x_0). \tag{1.3.22}$$

This yields

$$\begin{aligned}
\frac{\partial z^*(t^*, \tau^*, x^*)}{\partial t^*} &= \frac{\partial[x_0^*(t) + Q(t)(z(t, \tau, x) - x_0)]}{\partial(t + a)} \\
&= x_0^{*\prime}(t) + Q'(t)(z(t, \tau, x) - x_0) + Q(t)\frac{\partial z(t, \tau, x)}{\partial t} \\
&= x_0^{*\prime}(t) + Q'(t)Q(t)^\top(z^*(t^*, \tau^*, x^*) - x_0^*(t)) + Q(t)\frac{\partial z(t, \tau, x)}{\partial t}.
\end{aligned}$$

Substituting $t = \tau, t^* = \tau^*$ into this equality and using (1.3.4) and (1.3.5), we obtain:

$$\begin{aligned}
v^*(t^*, x^*) &= \left.\frac{\partial z^*(t^*, \tau^*, x^*)}{\partial t^*}\right|_{\tau^* = t^*} \\
&= x_0^{*\prime}(t) + Q'(t)Q(t)^\top(x^* - x_0^*(t)) + Q(t)v(t, x).
\end{aligned} \tag{1.3.23}$$

Therefore

$$\begin{aligned}
(\nabla v)^* &= \frac{\partial v^*(x^*, t^*)}{\partial x^*} \\
&= \frac{\partial(x_0^{*\prime}(t) + Q'(t)Q(t)^\top(x^* - x_0^*(t)) + Q(t)v(t, x))}{\partial x^*} \\
&= Q'(t)Q(t)^\top + \frac{\partial[Q(t)v(t, x)]}{\partial x}\frac{\partial x}{\partial x^*} \\
&= Q'(t)Q(t)^\top + Q(t)(\nabla v)\frac{\partial(x_0 + Q(t)^\top(x^* - x_0^*(t)))}{\partial x^*} \\
&= Q'(t)Q(t)^\top + Q(t)(\nabla v)Q(t)^\top.
\end{aligned}$$

Thus,

$$(\nabla v)^* = Q(t)(\nabla v)Q(t)^\top + Q'(t)Q(t)^\top. \tag{1.3.24}$$

Transpose this equality:

$$[(\nabla v)^*]^\top = Q(t)(\nabla v)^\top Q(t)^\top + Q(t)Q'(t)^\top. \tag{1.3.25}$$

Differentiating the identity $Q(t)Q(t)^\top = I$, we get

$$Q'(t)Q(t)^\top + Q(t)Q'(t)^\top = 0. \tag{1.3.26}$$

Taking into account (1.3.26), calculate the half-sum of equalities (1.3.24) and (1.3.25):

$$\begin{aligned}
\mathcal{E}^* &= \frac{1}{2}((\nabla v)^* + [(\nabla v)^*]^\top) \\
&= \frac{1}{2}[Q(t)(\nabla v)Q(t)^\top + Q'(t)Q(t)^\top + Q(t)(\nabla v)^\top Q(t)^\top + Q(t)Q'(t)^\top] \\
&= Q(t)\mathcal{E}Q(t)^\top.
\end{aligned}$$

Now consider the half-difference of (1.3.24) and (1.3.25):

$$\begin{aligned}
W^* &= \frac{1}{2}((\nabla v)^* - [(\nabla v)^*]^\top) \\
&= \frac{1}{2}[Q(t)(\nabla v)Q(t)^\top + Q'(t)Q(t)^\top - Q(t)(\nabla v)^\top Q(t)^\top - Q(t)Q'(t)^\top] \\
&= Q(t)WQ(t)^\top + Q'(t)Q(t)^\top. \qquad \square
\end{aligned}$$

1.3.6 Objective derivatives

According to the principle of material frame-indifference, a constitutive equation, as an expression of physical properties of a medium, should not change its form with a change of frame. Let us check whether this statement is valid for constitutive relations (1.3.1) and (1.3.12).

As we have already mentioned, the stress tensor T is frame-indifferent. Let us show that the deviatoric stress tensor is also frame-indifferent. We have:

$$\begin{aligned}
\sigma^* &= \mathrm{T}^* - \frac{1}{n}\,\mathrm{Tr}\,\mathrm{T}^* I = Q(t)\mathrm{T}Q(t)^\top - \frac{1}{n}\,\mathrm{Tr}(Q(t)\mathrm{T}Q(t)^\top)I \\
&= Q(t)\mathrm{T}Q(t)^\top - \frac{1}{n}\sum_{i,j,k=1}^{n} q_{ij}\mathrm{T}_{jk}q_{ik}I \\
&= Q(t)\mathrm{T}Q(t)^\top - \frac{1}{n}\sum_{j,k=1}^{n} [Q(t)^\top Q(t)]_{jk}\mathrm{T}_{jk}I
\end{aligned}$$

$$= Q(t)\mathrm{T}Q(t)^\top - \frac{1}{n}\sum_{j,k=1}^{n} I_{jk}\mathrm{T}_{jk} I$$

$$= Q(t)\mathrm{T}Q(t)^\top - \frac{1}{n}\operatorname{Tr}\mathrm{T}I = Q(t)\sigma Q(t)^\top. \tag{1.3.27}$$

This yields, in particular,

$$\sigma = Q(t)^\top \sigma^* Q(t); \tag{1.3.28}$$

and (1.3.20) implies:

$$\varepsilon = Q(t)^\top \varepsilon^* Q(t). \tag{1.3.29}$$

Substitute these representations into (1.3.1):

$$Q(t)^\top \sigma^* Q(t) + \lambda_1 \frac{\partial Q(t)^\top \sigma^* Q(t)}{\partial t} = 2\eta\big(Q(t)^\top \varepsilon^* Q(t) + \lambda_2 \frac{\partial Q(t)^\top \varepsilon^* Q(t)}{\partial t}\big).$$

Having applied $Q(t)$ from the left, and $Q(t)^\top$ from the right, we arrive at

$$\begin{aligned}\sigma^* &+ \lambda_1 \frac{\partial \sigma^*}{\partial t^*} + \lambda_1 Q(t)Q'(t)^\top \sigma^* Q(t)Q(t)^\top + \lambda_1 Q(t)Q(t)^\top \sigma^* Q'(t)Q(t)^\top \\ &= 2\eta\big(\varepsilon^* + \lambda_2 \frac{\partial \varepsilon^*}{\partial t^*} + \lambda_2 Q(t)Q'(t)^\top \varepsilon^* Q(t)Q(t)^\top \\ &\quad + \lambda_2 Q(t)Q(t)^\top \varepsilon^* Q'(t)Q(t)^\top\big).\end{aligned}$$

Now, use (1.3.26) and (1.3.15):

$$\begin{aligned}\sigma^* + \lambda_1 \frac{\partial \sigma^*}{\partial t^*} + \lambda_1 Q(t^*-a)Q'(t^*-a)^\top \sigma^* - \lambda_1 \sigma^* Q(t^*-a)Q'(t^*-a)^\top \\ = 2\eta\Big(\varepsilon^* + \lambda_2 \frac{\partial \varepsilon^*}{\partial t^*} + \lambda_2 Q(t^*-a)Q'(t^*-a)^\top \varepsilon^* - \lambda_2 \varepsilon^* Q(t^*-a)Q'(t^*-a)^\top\Big).\end{aligned}$$

Thus, after a change of observer additional terms appear in the constitutive relation (1.3.1). Similarly one shows that the same additional terms emerge in (1.3.12).

Remark 1.3.3. If a change of variables of the form (1.3.15) – (1.3.16) transforms inertial frames to inertial ones (see Remark 1.3.2), then $Q'(t)$ is identically zero, so (1.3.1) and (1.3.12) do not change their form.

Nevertheless, the appearing terms add up to null function provided the object which we substitute for the time derivative denoted by a point becomes more complicated.

Definition 1.3.1. An operator of the form

$$\frac{D\mathcal{T}(t,x)}{Dt} = \frac{d\mathcal{T}(t,x)}{dt} + G(\nabla v(t,x), \mathcal{T}(t,x)), \tag{1.3.30}$$

where G is a symmetric tensor-valued function of two tensor arguments, $\mathcal{T}(t,x)$ is a symmetric tensor-valued function, is called an *objective derivative*, if, for any change of frame (1.3.15) – (1.3.16), the equality

$$\frac{D^*\mathcal{T}^*}{Dt^*} = Q(t)\frac{D\mathcal{T}}{Dt}Q(t)^\top \tag{1.3.31}$$

holds for all frame-indifferent symmetric tensor-valued functions $\mathcal{T}(t,x)$.

Remark 1.3.4. The symbol $\frac{D^*}{Dt^*}$ denotes the representation of the operator $\frac{D}{Dt}$ in the new frame, i.e. the expression of the form

$$\frac{d^*\mathcal{T}^*}{dt^*} + G((\nabla v)^*, \mathcal{T}^*) = \frac{\partial \mathcal{T}^*}{\partial t^*} + \sum_{i=1}^n v_i^* \frac{\partial \mathcal{T}^*}{\partial x_i^*} + G((\nabla v)^*, \mathcal{T}^*).$$

The choice of the function G is realized according to various mechanical and experimental reasons.

Given an objective derivative $\frac{D}{Dt}$, we can pass from (1.2.17) to the constitutive relation (see Remark 1.3.5)

$$\sigma + \lambda_1 \frac{D\sigma}{Dt} = 2\eta\big(\mathcal{E} + \lambda_2 \frac{D\mathcal{E}}{Dt}\big). \tag{1.3.32}$$

This constitutive relation satisfies the principle of material frame-indifference. Really, substituting representations (1.3.28) and (1.3.29) into (1.3.32) and using property (1.3.31) of objective derivatives, we obtain:

$$Q(t)^\top \sigma^* Q(t) + \lambda_1 Q(t)^\top \frac{D^*\sigma^*}{Dt^*} Q(t) = 2\eta\Big(Q(t)^\top \mathcal{E}^* Q(t) + \lambda_2 Q(t)^\top \frac{D^*\mathcal{E}^*}{Dt^*} Q(t)\Big),$$

whence

$$\sigma^* + \lambda_1 \frac{D^*\sigma^*}{Dt^*} = 2\eta\left(\mathcal{E}^* + \lambda_2 \frac{D^*\mathcal{E}^*}{Dt^*}\right). \tag{1.3.33}$$

For various objective derivatives it can turn out to be very hard or impossible to express the tensor σ in terms of the strain characteristics explicitly from (1.3.32). Therefore, for description of motion, for example, of a homogeneous incompressible viscoelastic medium, it is necessary to consider system (1.1.10), (1.1.12), (1.3.32) with a large number of unknown functions.

1.3.7 Examples of objective derivatives

The simplest example of an objective derivative of a tensor is *Jaumann's derivative*:

$$\frac{D_0\mathcal{T}(t,x)}{Dt} = \frac{d\mathcal{T}(t,x)}{dt} + \mathcal{T}(t,x)W(t,x) - W(t,x)\mathcal{T}(t,x), \tag{1.3.34}$$

which is also called *corotational*. Let us show that it satisfies condition (1.3.31). We have, using (1.3.10), (1.3.21) and (1.3.26):

$$\frac{D_0^*\mathcal{T}^*}{Dt^*} = \frac{\partial}{\partial\tau^*}\mathcal{T}^*(\tau^*, z^*(\tau^*, t^*, x^*))\Big|_{\tau^*=t^*} + \mathcal{T}^*W^* - W^*\mathcal{T}^*$$
$$= \frac{\partial}{\partial\tau}\big[Q(\tau)\mathcal{T}(\tau, z(\tau,t,x))Q(\tau)^\top\big]\Big|_{\tau=t} + Q(t)\mathcal{T}Q(t)^\top\big(Q(t)WQ(t)^\top$$
$$+ Q'(t)Q(t)^\top\big) - \big(Q(t)WQ(t)^\top + Q'(t)Q(t)^\top\big)Q(t)\mathcal{T}Q(t)^\top$$
$$= Q'(t)\mathcal{T}Q(t)^\top + Q(t)\frac{d\mathcal{T}}{dt}Q(t)^\top + Q(t)\mathcal{T}Q'(t)^\top + Q(t)\mathcal{T}WQ(t)^\top$$
$$+ Q(t)\mathcal{T}Q^\top(t)Q'(t)Q(t)^\top - Q(t)W\mathcal{T}Q(t)^\top - Q'(t)Q(t)^\top Q(t)\mathcal{T}Q(t)^\top$$
$$= Q(t)\frac{D_0\mathcal{T}}{Dt}Q(t)^\top + Q'(t)\mathcal{T}Q(t)^\top + Q(t)\mathcal{T}Q'(t)^\top$$
$$- Q(t)\mathcal{T}Q'(t)^\top Q(t)Q(t)^\top - Q'(t)Q(t)^\top Q(t)\mathcal{T}Q(t)^\top = Q(t)\frac{D_0\mathcal{T}}{Dt}Q(t)^\top.$$

Below we show (see Remark 1.4.2) that every objective derivative may be represented as

$$\frac{D\mathcal{T}(t,x)}{Dt} = \frac{D_0\mathcal{T}(t,x)}{Dt} + G_1(\mathcal{E}(t,x), \mathcal{T}(t,x)), \tag{1.3.35}$$

where G_1 is a symmetric tensor-valued function of two symmetric tensor arguments. The sense of this representation is that every objective derivative is the sum of Jaumann's derivative and some expression which is independent of the vorticity tensor W. One can also give some information on the structure of G_1 (Corollary 1.4.2).

An elementary generalization of Jaumann's derivative is *Oldroyd's derivative*:

$$\frac{D_{\mathbf{a}}\mathcal{T}}{Dt} = \frac{D_0\mathcal{T}}{Dt} - \mathbf{a}(\mathcal{E}\mathcal{T} + \mathcal{T}\mathcal{E}). \tag{1.3.36}$$

This construction depends on a parameter $\mathbf{a} \in [-1, 1]$. For $\mathbf{a} = 0$ it becomes Jaumann's derivative.

For $\mathbf{a} = 1$, (1.3.36) is called the *upper-convected Maxwell derivative* (UCM), and, for $\mathbf{a} = -1$, the *lower-convected Maxwell derivative* (LCM).

A more general variant of objective derivative is due to Spriggs [7] and depends on three parameters $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$\frac{D_{\mathbf{abc}}\mathcal{T}}{Dt} = \frac{D_{\mathbf{a}}\mathcal{T}}{Dt} + \mathbf{b}\,\mathrm{Tr}(\mathcal{T}\mathcal{E})I + \mathbf{c}\,\mathrm{Tr}(\mathcal{T})\mathcal{E}, \tag{1.3.37}$$

where I is the unit tensor.

Remark 1.3.5. For generalizations of Jaumann's derivative, (1.3.32) may contradict (1.1.5) for the latter implies $\mathrm{Tr}\,\sigma = 0$. In this case, in (1.3.32), σ should not be understood as the deviatoric stress tensor but as a certain *extra-stress tensor*, which still satisfies (1.1.11) with some other p.

1.4 Nonlinear effects in viscous media

1.4.1 Nonlinear viscosity and viscoelasticity

The conception of viscoelastic medium is not unique for explanation and description of non-Newtonian behaviour of real fluids and the media close to fluids. There is another class of models: nonlinear-viscous media. Here it is supposed that the stress tensor at a moment t at a point x is a function of other characteristics of a medium, taken at the same point and at the same moment:

$$\mathrm{T}(t,x) = g_1(t,x,v(t,x),\nabla v(t,x),\rho(t,x)). \tag{1.4.1}$$

Then (1.1.5) implies that σ is also a function of these characteristics:

$$\sigma(t,x) = g_2(t,x,v(t,x),\nabla v(t,x),\rho(t,x)). \tag{1.4.2}$$

Remark 1.4.1. Obviously, constitutive relations (1.3.2), (1.3.13) and, all the more, (1.3.32) cannot be reduced to (1.4.2). The deviatoric stress tensor $\sigma(t,x)$ in these equations turns out to depend on characteristics of a medium at other points and at other moments. This property is a feature of *viscoelastic* models. In this sense, viscoelastic media and the models reducible to (1.4.2) may be considered as two opposite classes of models. On the other hand, nonlinear viscosity and viscoelasticity can be combined in one model (see Section 1.5).

1.4.2 Noll's theorem and the Stokes conjecture.

Relation (1.4.2) is the constitutive relation for the nonlinear-viscous medium. According to the principle of material frame-indifference, its structure should not depend on observer. This gives opportunity to simplify and specify the form of the constitutive relation (1.4.2). The first step in this direction is Noll's theorem.

Theorem 1.4.1. *If relation (1.4.2) satisfies the principle of material frame-indifference, then g_2 depends only on $\mathcal{E}$ and ρ:*

$$g_2(t,x,v,\nabla v,\rho) = g_3(\mathcal{E},\rho). \tag{1.4.3}$$

Proof. By virtue of the principle of material frame-indifference, for any change of observer (1.3.15) – (1.3.16) the form of the constitutive relation (1.4.2) remains constant. Note that the density ρ is frame-indifferent. Then, in view of representations (1.3.20), (1.3.21), (1.3.23) and (1.3.28), we get

$$\begin{aligned} g_2(t,x,v,\nabla v,\rho) &= Q^\top g_2(t^*,x^*,v^*,(\nabla v)^*,\rho^*)Q \\ &= Q^\top g_2(t+a, x_0^* + Q(x-x_0), x_0^{*\prime} + Q'Q^\top(x^*-x_0^*) \\ &\qquad + Qv, Q\mathcal{E}Q^\top + QWQ^\top + Q'Q^\top,\rho)Q. \end{aligned} \tag{1.4.4}$$

Let x_1 and t_1 be a fixed spatial point and a moment of time. Consider the tensor-valued function

$$Q(t) = e^{(t_1 - t)W(t_1, x_1)}, \tag{1.4.5}$$

where $W(t_1, x_1)$ is the vorticity tensor at (t_1, x_1).

Let us show that $Q(t)$ are orthogonal tensors. By a property of the exponential function, $Q'(t) = -Q(t)W(t_1, x_1)$. We have:

$$\begin{aligned}
Q(t_1)Q(t_1)^\top &= I,\\
(Q(t)Q(t)^\top)' &= Q'(t)Q(t)^\top + Q(t)Q'(t)^\top\\
&= -Q(t)WQ(t)^\top - Q(t)W^\top Q(t)^\top = 0
\end{aligned}$$

since W is skew-symmetric.

This yields

$$Q(t)Q(t)^\top \equiv I.$$

Observe that $Q(t_1) = I$, $Q'(t_1) = -W(t_1, x_1)$.

Put in (1.4.4): $t = t_1, x = x_1$, Q as in (1.4.5), and

$$x_0 = 0, \tag{1.4.6}$$

$$x_0^*(t) = -x_1 - (t - t_1)(v(t_1, x_1) - W(t_1, x_1)x_1), \tag{1.4.7}$$

$$a = -t_1. \tag{1.4.8}$$

We obtain:

$$\begin{aligned}
&g_2\big(t_1, x_1, v(t_1, x_1), \nabla v(t_1, x_1), \rho(t_1, x_1)\big)\\
&\quad= g_2\big(-a + a, -x_1 + x_1, -\big(v(t_1, x_1) - W(t_1, x_1)x_1\big) - W(t_1, x_1)x_1\\
&\qquad\quad + v(t_1, x_1), \mathcal{E}(t_1, x_1) + W(t_1, x_1) - W(t_1, x_1), \rho(t_1, x_1)\big)\\
&\quad= g_2(0, 0, 0, \mathcal{E}(t_1, x_1), \rho(t_1, x_1)).
\end{aligned}$$

Since t_1 and x_1 have been chosen arbitrarily, this continued equality implies the statement of the theorem. □

Remark 1.4.2. Let $\frac{D}{Dt}$ be an arbitrary objective derivative. Then, by (1.3.31), $\frac{D\mathcal{T}}{Dt}$ is frame-indifferent for any frame-indifferent tensor $\mathcal{T}(t, x)$. Since Jaumann's derivative is objective, $\frac{D\mathcal{T}}{Dt} - \frac{D_0\mathcal{T}}{Dt}$ is also frame-indifferent. But by (1.3.30) and (1.3.34) this expression is equal to $G(\nabla v, \mathcal{T}) - \mathcal{T}W + W\mathcal{T}$. Denote it by $G_1(\nabla v, \mathcal{T})$. Take arbitrary t_1 and x_1, and define Q by formula (1.4.5). Since $G_1(\nabla v, \mathcal{T})$ is frame-indifferent, just as in the proof of Noll's theorem we obtain:

$$\begin{aligned}
G_1(\nabla v, \mathcal{T}) &= Q^\top G_1\big((\nabla v)^*, \mathcal{T}^*\big)Q\\
&= Q^\top G_1\big(Q\mathcal{E}Q^\top + QWQ^\top + Q'Q^\top, Q\mathcal{T}Q^\top\big)Q.
\end{aligned} \tag{1.4.9}$$

At the moment t_1 and at the point x_1 one has:

$$G_1\big(\nabla v(t_1,x_1),\mathcal{T}(t_1,x_1)\big) = G_1\big(\mathcal{E}(t_1,x_1),\mathcal{T}(t_1,x_1)\big),$$

This implies representation (1.3.35).

Now, by Noll's theorem, relation (1.4.2) can be rewritten as:

$$\sigma(t,x) = g_3\big(\mathcal{E}(t,x),\rho(t,x)\big).$$

In the case when the deviatoric stress tensor does not depend on density (for example, the density is constant), we simply have:

$$\sigma(t,x) = g_3\big(\mathcal{E}(t,x)\big). \tag{1.4.10}$$

This relation is called the Stokes conjecture.

1.4.3 The Wang and Rivlin–Ericksen theorems

The presence of frame-indifference gives an opportunity to get some information on the function g_3 from (1.4.10) as well as on the function G_1 from (1.3.35). The main tool here is the following theorem, which is a particular case of Wang's theorem ([80], p. 215, see also [79], p. 197).

Theorem 1.4.2. *Let $n = 3$ and let a symmetric tensor-valued function $\zeta(\mathcal{T}_1,\mathcal{T}_2)$ of two symmetric tensor arguments satisfy the condition*

$$\zeta(\mathcal{T}_1,\mathcal{T}_2) = Q^\top \zeta(Q\mathcal{T}_1Q^\top, Q\mathcal{T}_2Q^\top)Q, \tag{1.4.11}$$

for each orthogonal tensor Q and all symmetric tensors $\mathcal{T}_1,\mathcal{T}_2$. Then ζ can be represented as follows:

$$\begin{aligned}\zeta(\mathcal{T}_1,\mathcal{T}_2) = \alpha_0 I &+ \alpha_1\mathcal{T}_1 + \alpha_2\mathcal{T}_1^2 + \alpha_3\mathcal{T}_2 + \alpha_4\mathcal{T}_2^2\\ &+ \alpha_5(\mathcal{T}_1\mathcal{T}_2 + \mathcal{T}_2\mathcal{T}_1) + \alpha_6(\mathcal{T}_1^2\mathcal{T}_2 + \mathcal{T}_2\mathcal{T}_1^2)\\ &+ \alpha_7(\mathcal{T}_1\mathcal{T}_2^2 + \mathcal{T}_2^2\mathcal{T}_1) + \alpha_8(\mathcal{T}_1^2\mathcal{T}_2^2 + \mathcal{T}_2^2\mathcal{T}_1^2)\end{aligned} \tag{1.4.12}$$

where each α_j is a scalar function of ten scalar arguments:

$$\alpha_j = \alpha_j\big(\operatorname{Tr}\mathcal{T}_1, \operatorname{Tr}(\mathcal{T}_1^2), \operatorname{Tr}(\mathcal{T}_1^3), \operatorname{Tr}(\mathcal{T}_2), \operatorname{Tr}(\mathcal{T}_2^2), \operatorname{Tr}(\mathcal{T}_2^3), \operatorname{Tr}(\mathcal{T}_1\mathcal{T}_2),$$
$$\operatorname{Tr}(\mathcal{T}_1^2\mathcal{T}_2), \operatorname{Tr}(\mathcal{T}_1\mathcal{T}_2^2), \operatorname{Tr}(\mathcal{T}_1^2\mathcal{T}_2^2)\big).$$

Remark 1.4.3. This particular case of Wang's theorem has no concern with the controversy related to the proof of his representation theorems (see [57], [81]).

The following statement is a special case of the well-known Rivlin–Ericksen theorem.

Corollary 1.4.1. *For $n = 3$, the function g_3 from (1.4.10) may be represented as follows:*

$$g_3(\mathcal{E}) = \varphi_0(I_1, I_2, I_3)I + \varphi_1(I_1, I_2, I_3)\mathcal{E} + \varphi_2(I_1, I_2, I_3)\mathcal{E}^2, \tag{1.4.13}$$

where

$$I_1 = \operatorname{Tr}\mathcal{E} = \operatorname{div} v, I_2 = \operatorname{Tr}\mathcal{E}^2 = \sum_{i,j=1}^{n} \mathcal{E}_{ij}^2, I_3 = \det\mathcal{E},$$

and $\varphi_0, \varphi_1, \varphi_2$ are scalar functions of three scalar arguments.

Proof. Observe that (1.4.3) and (1.4.4) yield:

$$g_3(\mathcal{E}) = Q^\top g_3(\mathcal{E}^*)Q,$$

and (1.3.20) implies

$$g_3(\mathcal{E}) = Q^\top g_3(Q\mathcal{E}Q^\top)Q, \tag{1.4.14}$$

for any orthogonal tensor Q.

Without loss of generality we may assume that $g_3(\mathcal{T}_1)$ is defined for all symmetric tensors $\mathcal{T}_1$ (for example, put $g_3(\mathcal{T}_1) = 0$ for all $\mathcal{T}_1$ which cannot be represented as $Q\mathcal{E}Q^\top$).

It is easy to check that

$$\operatorname{Tr}(\mathcal{T}_1^3) = -\frac{1}{2}(\operatorname{Tr}\mathcal{T}_1)^3 + \frac{3}{2}\operatorname{Tr}\mathcal{T}_1 \operatorname{Tr}(\mathcal{T}_1^2) + 3\det\mathcal{T}_1. \tag{1.4.15}$$

It remains to apply Theorem 1.4.2 with $\zeta(\mathcal{T}_1, \mathcal{T}_2) = g_3(\mathcal{T}_1)$ and to use representation (1.4.12) with $\mathcal{T}_2 = I$. □

For a homogeneous incompressible medium one has $I_1 = \operatorname{div} v = 0$. Therefore, in this case, (1.4.13) has a simpler form:

$$g_3(\mathcal{E}) = \varphi_0(I_2, I_3)I + \varphi_1(I_2, I_3)\mathcal{E} + \varphi_2(I_2, I_3)\mathcal{E}^2. \tag{1.4.16}$$

It yields

$$\operatorname{Div}\sigma = \operatorname{Div}(\varphi_0 I + \varphi_1\mathcal{E} + \varphi_2\mathcal{E}^2) = \nabla\varphi_0 + \operatorname{Div}(\varphi_1\mathcal{E} + \varphi_2\mathcal{E}^2).$$

To obtain the equation of motion, substitute it into (1.1.12):

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \nabla\varphi_0 - \operatorname{Div}(\varphi_1\mathcal{E} + \varphi_2\mathcal{E}^2) + \operatorname{grad} p = f. \tag{1.4.17}$$

Let $\widetilde{p}(t,x) = p(t,x) - \varphi_0(I_2(t,x), I_3(t,x))$. We have obtained the general equation of motion for a homogeneous incompressible nonlinear-viscous medium:

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \mathrm{Div}\,(\varphi_1(I_2, I_3)\mathcal{E} + \varphi_2(I_2, I_3)\mathcal{E}^2) + \nabla\widetilde{p} = f. \tag{1.4.18}$$

Remark 1.4.4. This shows that $\widetilde{p}$ has the same "essence" as the hydrostatic pressure p, so without loss of generality of the model one can assume that $\varphi_0 \equiv 0$. However, it can be applied to the equation of motion, which does not contain σ, and it cannot be done directly for (1.4.13), because it may violate the assertion $\mathrm{Tr}\,\sigma = 0$, which follows from (1.1.5).

Due to representation (1.4.9), Theorem 1.4.2 is applicable to the function G_1 from (1.3.35). Thus, we have

Corollary 1.4.2. *For $n = 3$, every objective derivative may be represented as*

$$\frac{D\mathcal{T}(t,x)}{Dt} = \frac{D_0\mathcal{T}(t,x)}{Dt} + \alpha_0 I + \alpha_1\mathcal{E} + \alpha_2\mathcal{E}^2 + \alpha_3\mathcal{T} + \alpha_4\mathcal{T}^2 + \alpha_5(\mathcal{E}\mathcal{T} + \mathcal{T}\mathcal{E})$$
$$+\alpha_6(\mathcal{E}^2\mathcal{T} + \mathcal{T}\mathcal{E}^2) + \alpha_7(\mathcal{E}\mathcal{T}^2 + \mathcal{T}^2\mathcal{E}) + \alpha_8(\mathcal{E}^2\mathcal{T}^2 + \mathcal{T}^2\mathcal{E}^2) \tag{1.4.19}$$

where each α_j is a scalar function of ten scalar arguments:

$$\begin{aligned}\alpha_j = \alpha_j(&\mathrm{Tr}\,\mathcal{E}, \mathrm{Tr}(\mathcal{E}^2), \mathrm{Tr}(\mathcal{E}^3), \mathrm{Tr}(\mathcal{T}), \mathrm{Tr}(\mathcal{T}^2), \mathrm{Tr}(\mathcal{T}^3),\\ &\mathrm{Tr}(\mathcal{E}\mathcal{T}), \mathrm{Tr}(\mathcal{E}^2\mathcal{T}), \mathrm{Tr}(\mathcal{E}\mathcal{T}^2), \mathrm{Tr}(\mathcal{E}^2\mathcal{T}^2)).\end{aligned} \tag{1.4.20}$$

Remark 1.4.5. The results of this subsection are also true for $n = 2$. Their derivation from the given three-dimensional results is a good exercise in matrix theory.

1.4.4 Oldroyd's method. Models of Prandtl and Eyring

The functions $\varphi_0, \varphi_1, \varphi_2$ in (1.4.13) are to be determined experimentally. In order to simplify the construction of these functions, one may apply some reduction to a one-dimensional model (as in the case of the method of mechanical models). For example, Oldroyd [44] suggested the following procedure. Using simple flows, one finds a one-dimensional connection between the stress $\sigma_1 = \sigma\vec{n}$ and the strain velocity $\mathcal{E}_1 = \mathcal{E}\vec{n}$ in a certain direction $\vec{n}$ (for example, in the longitudinal direction, for the pipe flow):

$$\sigma_1 = \psi(\mathcal{E}_1). \tag{1.4.21}$$

It is assumed that $\psi(0) = 0$.

Let $\psi_1(\mathcal{E}_1) = \frac{\psi(\mathcal{E}_1)}{\mathcal{E}_1}$ ($\mathcal{E}_1 \neq 0$). We have:

$$\sigma_1 = \psi_1(\mathcal{E}_1)\mathcal{E}_1 \quad (\mathcal{E}_1 \neq 0). \tag{1.4.22}$$

But since $\psi(0) = 0$,

$$\sigma_1 = 0 \quad (\mathcal{E}_1 = 0). \tag{1.4.23}$$

In order to pass to the n-dimensional situation, one substitutes σ for σ_1, and $\mathcal{E}$ for $\mathcal{E}_1$. However, since ψ_1 is a function of scalar argument, one has to substitute some scalar describing the strain velocity for this argument. The simplest variant is the Euclidean norm of the strain velocity tensor: $\sqrt{\sum_{i,j=1}^{n} \mathcal{E}_{ij}^2} = \sqrt{I_2}$ (for simplicity, here we assume incompressibility).

We obtain the following constitutive relation:

$$\sigma = \psi_1(\sqrt{I_2})\mathcal{E} \quad (\mathcal{E} \neq 0); \tag{1.4.24}$$

$$\sigma = 0 \quad (\mathcal{E} = 0). \tag{1.4.25}$$

In order to get a constitutive law of the same form as (1.4.13), put

$$\varphi_0 \equiv \varphi_2 \equiv 0, \quad \varphi_1(I_2, I_3) = \psi_1(\sqrt{I_2}) \quad (I_2 \neq 0), \tag{1.4.26}$$
$$\varphi_1(I_2, I_3) = 0 \quad (I_2 = 0).$$

Let us consider the application of this method to the models of Prandtl and Eyring.

In this subsection A, B, C mean positive constants, which may be measured for real fluids. The one-dimensional Prandtl's model [83] looks like

$$\sigma_1 = A \arcsin\left(\frac{\mathcal{E}_1}{C}\right), \tag{1.4.27}$$

where σ_1 and $\mathcal{E}_1$ are the stress and the strain velocity. "Extracting viscosity", we get

$$\sigma_1 = \frac{A \arcsin\left(\frac{\mathcal{E}_1}{C}\right)}{\mathcal{E}_1}\mathcal{E}_1. \tag{1.4.28}$$

Passing to the n-dimensional model, we have:

$$\sigma = \frac{A \arcsin\left(\frac{\sqrt{I_2}}{C}\right)}{\sqrt{I_2}}\mathcal{E}, \tag{1.4.29}$$

together with (1.4.25).

Eyring's model [83] has (in the one-dimensional situation) the form:

$$\sigma_1 = \frac{\mathcal{E}_1}{B} + C \sin\left(\frac{\sigma_1}{A}\right). \tag{1.4.30}$$

We have :

$$\mathcal{E}_1 = B\sigma_1 - BC \sin\left(\frac{\sigma_1}{A}\right). \tag{1.4.31}$$

Consider the case $C < A$. Then the function in the right-hand side of (1.4.31) is invertible on a neighborhood of the origin. Denote the inverse function by ς. Evidently, $\varsigma(0) = 0$. Then

$$\sigma_1 = \varsigma(\mathcal{E}_1) \tag{1.4.32}$$

for small strain velocities.

This implies the n-dimensional constitutive law:

$$\sigma = \frac{\varsigma(\sqrt{I_2})}{\sqrt{I_2}}\mathcal{E}, \tag{1.4.33}$$

together with (1.4.25).

Remark 1.4.6. The functions $\frac{A \arcsin\left(\frac{\sqrt{I_2}}{C}\right)}{\sqrt{I_2}}$ and $\frac{\varsigma(\sqrt{I_2})}{\sqrt{I_2}}$ are even (as functions of I_2), so they are in fact smooth functions of I_2.

1.5 Combined models of nonlinear viscoelastic media

1.5.1 Nonlinear differential constitutive relations

In this section we assume that $n = 2$ or 3.

Rewrite the Jeffreys relation (1.3.32) using (1.3.35):

$$\sigma + \lambda_1\Big(\frac{D_0\sigma}{Dt} + G_1(\mathcal{E}, \sigma)\Big) = 2\eta\Big(\mathcal{E} + \lambda_2\Big(\frac{D_0\mathcal{E}}{Dt} + G_1(\mathcal{E}, \mathcal{E})\Big)\Big). \tag{1.5.1}$$

Denote $\mu_1 = \eta\dfrac{\lambda_2}{\lambda_1}$, $\eta_1 = \eta - \mu_1$, $\tau = \sigma - 2\mu_1\mathcal{E}$. Then (1.5.1) implies

$$\tau + \lambda_1\frac{D_0\tau}{Dt} + \lambda_1 G_1(\mathcal{E}, \tau + 2\mu_1\mathcal{E}) - 2\eta\lambda_2 G_1(\mathcal{E}, \mathcal{E}) = 2\eta_1\mathcal{E}. \tag{1.5.2}$$

Denote $\beta(\tau, \mathcal{E}) = \lambda_1 G_1(\mathcal{E}, \tau + 2\mu_1\mathcal{E}) - 2\eta\lambda_2 G_1(\mathcal{E}, \mathcal{E})$.

We have:

$$\tau + \lambda_1\frac{D_0\tau}{Dt} + \beta(\tau, \mathcal{E}) = 2\eta_1\mathcal{E}. \tag{1.5.3}$$

As in the case with Corollary 1.4.2, Theorem 1.4.2 is applicable to β. Hence, $\beta(\tau, \mathcal{E})$ may be represented as

$$\begin{aligned} &\alpha_0 I + \alpha_1\mathcal{E} + \alpha_2\mathcal{E}^2 + \alpha_3\tau + \alpha_4\tau^2 + \alpha_5(\mathcal{E}\tau + \tau\mathcal{E}) \\ &\quad + \alpha_6(\mathcal{E}^2\tau + \tau\mathcal{E}^2) + \alpha_7(\mathcal{E}\tau^2 + \tau^2\mathcal{E}) + \alpha_8(\mathcal{E}^2\tau^2 + \tau^2\mathcal{E}^2) \end{aligned} \tag{1.5.4}$$

where each α_j is a scalar function of ten scalar arguments:

$$\alpha_j = \alpha_j\big(\operatorname{Tr}\mathcal{E}, \operatorname{Tr}(\mathcal{E}^2), \operatorname{Tr}(\mathcal{E}^3), \operatorname{Tr}(\tau), \operatorname{Tr}(\tau^2),\\ \operatorname{Tr}(\tau^3), \operatorname{Tr}(\mathcal{E}\tau), \operatorname{Tr}(\mathcal{E}^2\tau), \operatorname{Tr}(\mathcal{E}\tau^2), \operatorname{Tr}(\mathcal{E}^2\tau^2)\big).$$

Relation (1.5.3) is a widespread form of constitutive equations for viscoelastic medium. Here are some examples of the function $\beta(\tau, \mathcal{E})$ (see [21, 22]).

The Giesekus model has

$$\beta(\tau, \mathcal{E}) = -\lambda_1(\tau\mathcal{E} + \mathcal{E}\tau) + \alpha\tau^2. \tag{1.5.5}$$

The Phan-Thien and Tanner model corresponds to

$$\beta(\tau, \mathcal{E}) = -\lambda_1(\tau\mathcal{E} + \mathcal{E}\tau) + \alpha\tau \operatorname{Tr}\ \tau. \tag{1.5.6}$$

In both cases α is a constant.

Larson's generalization of these two models has

$$\beta(\tau, \mathcal{E}) = -\lambda_1(\tau\mathcal{E} + \mathcal{E}\tau) + \alpha_1\tau^2 + \alpha_2\tau, \tag{1.5.7}$$

where α_1 and α_2 are scalar functions of $\operatorname{Tr}\ \tau$ and $\det\tau$, and $\alpha_2(0,0) = 0$.

The Larson model (another one) corresponds to

$$\beta(\tau, \mathcal{E}) = -\lambda_1(\tau\mathcal{E} + \mathcal{E}\tau) + 2\alpha_3 \operatorname{Tr}(\tau\mathcal{E})(\tau + I) \tag{1.5.8}$$

where α_3 is a scalar function of $\operatorname{Tr}\ \tau$.

The Oldroyd "8 constants" model has

$$\beta(\tau, \mathcal{E}) = -a\lambda_1(\tau\mathcal{E} + \mathcal{E}\tau) + b\operatorname{Tr}(\tau\mathcal{E})I + c\operatorname{Tr}(\tau)\mathcal{E} + c_1\mathcal{E}^2 + c_2 I_2 I, \tag{1.5.9}$$

where a, b, c, c_1, c_2 are constants. For $c_1 = c_2 = 0$ it is equivalent to Jeffreys' model with Spriggs' derivative (1.3.37).

Remark 1.5.1. Using the notation $\tau = \sigma - 2\mu_1\mathcal{E}$ and (1.1.13), one rewrites the equation of motion (1.1.12) as

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \mu_1\Delta v - \operatorname{Div}\ \tau + \operatorname{grad} p = f. \tag{1.5.10}$$

Remark 1.5.2. There is an interesting generalization of Jeffreys' model. It is assumed that the relaxation time λ_1 and the viscosity (more precisely, the viscosity of Maxwell's constituent $(1 - \lambda_2/\lambda_1)\eta$) depend on the Euclidean norm I_2 of the strain velocity tensor (compare with (1.4.26)). Such models are called the models of the White–Metzner type [22]. Mathematical investigation of the boundary value problems for equations of motion of these media was carried out by Hakim (see [26]).

1.5.2 Combined models

In order to get wider and more realistic classes of constitutive relations one can use the following simple heuristic argument inspired by the method of mechanical models.

Assume that we have m various models, each one with its own constitutive relation, which gives a connection between the deviatoric stress tensor σ and deformation characteristics. Let M_j, $j = 1, \dots, m$, be the "symbolical representations" (compare with Section 1.2.1) for each of these models. Consider the combined model $M_1|M_2|\dots|M_m$. Roughly speaking, this combined model describes the mixture of given m mediums. Since in the case of parallel connection the stresses perceived by the elements are summarized, and the deformation is common, we have the following constitutive law for the combined model:

$$\sigma = \sum_{k=1}^{m} \sigma_{M_k}. \tag{1.5.11}$$

Let us apply this method to the general models that we have: nonlinear-viscous fluid satisfying the Stokes conjecture (1.4.10) and the Jeffreys model (1.3.32). Observe that combining of several relations like (1.4.10) cannot produce something essentially new. Thus, we combine several Jeffreys' models (1.3.32), each one with its own parameters, with one nonlinear-viscous model (1.4.10), and obtain the constitutive law:

$$\sigma = \sum_{k=0}^{r} \sigma_k, \tag{1.5.12}$$

$$\sigma_0 = \varphi_0(I_1, I_2, I_3)I + \varphi_1(I_1, I_2, I_3)\mathcal{E} + \varphi_2(I_1, I_2, I_3)\mathcal{E}^2, \tag{1.5.13}$$

$$\begin{aligned} &\sigma_k + \lambda_k\Big(\frac{D_0\sigma_k}{Dt} + G_k(\mathcal{E}, \sigma_k)\Big) \\ &\quad = 2\nu_k\Big(\mathcal{E} + \lambda_{2,k}\Big(\frac{D_0\mathcal{E}}{Dt} + G_k(\mathcal{E}, \mathcal{E})\Big)\Big), \quad k = 1, \dots, r. \end{aligned} \tag{1.5.14}$$

Here ν_k are viscosities, λ_k are relaxation times, and $\lambda_{2,k}$ are retardation times.

Assume that the medium is homogeneous and incompressible. Then the argument I_1 disappears in (1.5.13), as in (1.4.16).

Introduce the notations: $\mu_k = \nu_k \dfrac{\lambda_{2,k}}{\lambda_k}$, $\eta_k = \nu_k - \mu_k$, $\tau^k = \sigma_k - 2\mu_k\mathcal{E}$, $\beta_k(\tau, \mathcal{E}) = \lambda_k G_k(\mathcal{E}, \tau^k + 2\mu_k\mathcal{E}) - 2\nu_k\lambda_{2,k}G_k(\mathcal{E}, \mathcal{E})$, $k = 1, \dots, r$.

Then, as in the previous subsection, we can rewrite (1.5.14) as

$$\tau^k + \lambda_k\frac{D_0\tau^k}{Dt} + \beta_k(\tau^k, \mathcal{E}) = 2\eta_k\mathcal{E} \tag{1.5.15}$$

where

$$\begin{aligned}\beta_k(\tau, \mathcal{E}) = {} & \alpha_0^k I + \alpha_1 \mathcal{E} + \alpha_2^k \mathcal{E}^2 + \alpha_3^k \tau + \alpha_4^k \tau^2 + \alpha_5^k (\mathcal{E}\tau + \tau\mathcal{E}) \\ & + \alpha_6^k (\mathcal{E}^2\tau + \tau\mathcal{E}^2) + \alpha_7^k (\mathcal{E}\tau^2 + \tau^2\mathcal{E}) + \alpha_8^k (\mathcal{E}^2\tau^2 + \tau^2\mathcal{E}^2).\end{aligned} \tag{1.5.16}$$

Each α_j^k is a scalar function of nine scalar arguments:

$$\begin{aligned}\alpha_j^k = \alpha_j^k \big(& \mathrm{Tr}(\mathcal{E}^2), \mathrm{Tr}(\mathcal{E}^3), \mathrm{Tr}(\tau), \mathrm{Tr}(\tau^2), \\ & \mathrm{Tr}(\tau^3), \mathrm{Tr}(\mathcal{E}\tau), \mathrm{Tr}(\mathcal{E}^2\tau), \mathrm{Tr}(\mathcal{E}\tau^2), \mathrm{Tr}(\mathcal{E}^2\tau^2)\big).\end{aligned} \tag{1.5.17}$$

Let $\widetilde{p} = p - \varphi_0(I_2, I_3), \widetilde{\varphi}_1 = \varphi_1 + 2 \sum_{j=1}^{r} \mu_k$ and

$$\tau^0 = \Psi(\mathcal{E}(u)) = \widetilde{\varphi}_1(I_2, I_3)\mathcal{E} + \varphi_2(I_2, I_3)\mathcal{E}^2. \tag{1.5.18}$$

Then (1.1.11) and (1.5.12) yield:

$$\mathrm{T} = -\widetilde{p} I + \sum_{k=0}^{r} \tau^k. \tag{1.5.19}$$

The system (1.5.15) – (1.5.19) is the general combined constitutive law for homogeneous incompressible nonlinear viscoelastic medium. For description of motion, it must be considered together with (1.1.9) and (1.1.10).

Chapter 2

Basic function spaces. Embedding and compactness theorems

2.1 Function spaces and embeddings

2.1.1 Lebesgue and Sobolev spaces

Let $n \in \mathbb{N}$ and $\mathbb{R}^n$ be the arithmetical n-dimensional space. In this book open sets in $\mathbb{R}^n$ are often called *domains*.

Let $\Omega \subset \mathbb{R}^n$ be a domain. Denote by $L_p(\Omega)$, $p \in \mathbb{R}$, $1 \leq p < \infty$, the space of all Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ for which the function $|u(x)|^p$ is Lebesgue integrable. Of course, if two measurable or integrable functions coincide almost everywhere in Ω, they are considered to be equal. The norm in the space $L_p(\Omega)$, $1 \leq p < \infty$, is defined as

$$\|u\|_{L_p(\Omega)} = \Big(\int_\Omega |u(x)|^p \, dx \Big)^{\frac{1}{p}}.$$

In the case $p = \infty$ the space $L_p(\Omega)$ is defined as the space of all Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ which are bounded by a constant almost everywhere on Ω. The norm in this space is given by the formula

$$\|u\|_{L_\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|.$$

A well-known property of L_p-norms is Hölder's inequality: for

$$1 \leq p, p_1, p_2 \leq \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \psi_1 \in L_{p_1}(\Omega), \quad \psi_2 \in L_{p_2}(\Omega),$$

one has that the pointwise product $\psi_1 \psi_2 \in L_p(\Omega)$, and

$$\|\psi_1 \psi_2\|_{L_p(\Omega)} \leq \|\psi_1\|_{L_{p_1}(\Omega)} \|\psi_2\|_{L_{p_2}(\Omega)}. \tag{2.1.1}$$

Here it is supposed $\frac{1}{\infty} = 0$.

For $u, v \in L_1(\Omega)$ such that $uv \in L_1(\Omega)$, introduce the bilinear form

$$(u, v) = \int_\Omega u(x) v(x) \, dx.$$

Hereafter we write simply $\|u\|$ for $\|u\|_{L_2(\Omega)}$. This norm becomes Euclidean if we define the scalar product in $L_2(\Omega)$ by the formula

$$(u, v)_{L_2(\Omega)} = (u, v).$$

Denote by $C(\overline{\Omega})$ the set of continuous functions $u : \overline{\Omega} \to \mathbb{R}$ (here $\overline{\Omega}$ is the closure of Ω). If Ω is bounded, $C(\overline{\Omega})$ is a Banach space with the norm

$$\|u\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |u(x)|.$$

For $u \in C(\overline{\Omega})$,

$$\|u\|_{C(\overline{\Omega})} = \|u\|_{L_\infty(\Omega)}.$$

Denote by $C^1(\overline{\Omega})$ the set of continuously differentiable functions $u : \overline{\Omega} \to \mathbb{R}$ (more precisely, they must be defined in a neighbourhood of Ω in order to deal with the derivatives on the boundary). If Ω is bounded, it is a Banach space with the norm

$$\|u\|_{C^1(\overline{\Omega})} = \|u\|_{C(\overline{\Omega})} + \sum_{i=1}^{n} \|\frac{\partial u}{\partial x_i}\|_{C(\overline{\Omega})}.$$

If α is a *multi-index*, i.e. $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, where $k, \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N} \cup \{0\}$, and if $u \in L_p(\Omega)$, then $D^\alpha u$ stands for the generalized (Sobolev) partial derivative of the function $u(x)$, $x \in \Omega$:

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_k^{\alpha_k}}.$$

Here $|\alpha| = \sum_{j=1}^{k} \alpha_j$.

Denote by $W_p^r(\Omega)$, $1 \le p \le \infty, r \in \mathbb{N}$ or $r = 0$, the Sobolev space of such functions $u \in L_p(\Omega)$ that $D^\alpha u \in L_p(\Omega)$ for $|\alpha| \le r$. The norm in this space is

$$\|u\|_{W_p^r(\Omega)} = \Big(\sum_{|\alpha|\le r} \|D^\alpha u\|_{L_p(\Omega)}^p \Big)^{\frac{1}{p}}, \quad p < \infty;$$

$$\|u\|_{W_p^r(\Omega)} = \max_{|\alpha|\le r} \|D^\alpha u\|_{L_\infty(\Omega)}, \quad p = \infty.$$

The norm in $W_2^r(\Omega)$ is Euclidean, and the corresponding scalar product can be defined as

$$(u, v)_{W_2^r(\Omega)} = \sum_{|\alpha|\le r} (D^\alpha u, D^\alpha v).$$

Hereafter we often write $(u, v)_r$ for $(u, v)_{W_2^r(\Omega)}$ and $\|u\|_r$ for $\|u\|_{W_2^r(\Omega)}$. Note that $(u, v)_0 = (u, v)$ and $\|u\|_0 = \|u\|$.

Denote by $C_0^\infty(\Omega)$ the set of smooth scalar functions with compact support in Ω. Denote by $\overset{\circ}{W}{}_p^r(\Omega)$ the subspace of $W_p^r(\Omega)$ which is the closure of $C_0^\infty(\Omega)$.

Now we can define $W_p^{-r}(\Omega)$, $1 < p \leq \infty, r \in \mathbb{N}$, as the dual space of $\overset{\circ}{W}{}_q^r(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2.1.1. We recall that the dual space X^* of a real normed space X consists of all continuous linear functionals $\varphi : X \to \mathbb{R}$. In this book we denote the value of a functional $\varphi \in X^*$ on an element $w \in X$ by

$$\langle \varphi, w \rangle_{X^* \times X}$$

or simply by

$$\langle \varphi, w \rangle$$

(it is called the "bra–ket" notation). The norm in the space X^* is defined as

$$\|\varphi\|_{X^*} = \sup_{\|w\|_X = 1} |\langle \varphi, w \rangle|.$$

The $*$-*weak* topology on X^* may be set as follows: $\varphi_m \to \varphi_0$ $*$-weakly if $\langle \varphi_m, w \rangle \to \langle \varphi_0, w \rangle$ for all $w \in X$. If X is separable, any bounded set in X^* is relatively compact in this $*$-weak topology.

We recall also that X is called *reflexive* if every linear continuous functional on X^* may be represented in the form

$$\langle \cdot, w \rangle,$$

where w is some element of X. Every bounded set in a reflexive space X is relatively compact in the weak topology of X.

The spaces $W_2^r(\Omega)$ and $\overset{\circ}{W}{}_2^r(\Omega)$ are also denoted $H^r(\Omega)$ and $H_0^r(\Omega)$, respectively. We shall use the generalization of $H^r(\Omega)$, the Sobolev–Slobodetskii space $H^s(\Omega)$ (or $W_2^s(\Omega)$), where $s \in \mathbb{R}\backslash\mathbb{N}$, $s > 0$. Let the brackets $[\cdot]$ and $\{\cdot\}$ denote the integer part and the fractional part of a number, respectively. Then $H^s(\Omega)$ consists of the functions $u \in H^{[s]}(\Omega)$ for which the norm

$$\|u\|_{H^s(\Omega)} = \left(\|u\|_{[s]}^2 + \sum_{|\alpha| = [s]} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{\|x - y\|_{\mathbb{R}^n}^{n+2\{s\}}} \, dx dy \right)^{1/2}$$

is finite.

Remark 2.1.2. Another approach to the definition of $H^s(\Omega)$ in the case $\Omega = \mathbb{R}^n$ will be described in Section 5.2.2.

Denote by $H_0^s(\Omega)$ the subspace of $H^s(\Omega)$ which is the closure of $C_0^\infty(\Omega)$.

The Sobolev and Sobolev–Slobodetskii spaces are Banach spaces. They are reflexive for $1 < p < \infty$ and separable for $p \neq \infty$.

Following L. Schwartz, denote by $\mathcal{D}(\Omega)$ the set $C_0^\infty(\Omega)$ equipped with the following topology: a sequence $\psi_k \in \mathcal{D}(\Omega)$ converges to $\psi_0 \in \mathcal{D}(\Omega)$ if

i) there exists a compact set $\omega \subset \Omega$ such that $\psi_k(x) = \psi_0(x)$ for every natural k and any $x \in \Omega \backslash \omega$,

ii) for every multi-index α

$$\sup_{x \in \Omega} |D^\alpha(\psi_k(x) - \psi_0(x))| \to 0.$$

Let $\mathcal{D}'(\Omega)$ be the set of continuous linear functionals on the space $\mathcal{D}(\Omega)$. The elements of the space $\mathcal{D}'(\Omega)$ are called *generalized functions, scalar distributions* or simply *distributions*. The value of a functional $\varphi \in \mathcal{D}'(\Omega)$ on an element $\psi \in \mathcal{D}(\Omega)$ is denoted as $\langle \varphi, \psi \rangle$ (the "bra–ket" notation).

The topology on $\mathcal{D}'(\Omega)$ can be set as follows: a sequence $u_k \in \mathcal{D}'(\Omega)$ converges to $u_0 \in \mathcal{D}'(\Omega)$ if for any $\psi \in \mathcal{D}(\Omega)$ one has

$$\langle u_k, \psi \rangle \to \langle u_0, \psi \rangle.$$

We recall the following chain of inclusions:

$$\mathcal{D}(\Omega) \subset W_p^r(\Omega) \subset L_p(\Omega) \subset W_p^{-r}(\Omega) \subset \mathcal{D}'(\Omega) \tag{2.1.2}$$

where $1 < p \leq \infty, r \in \mathbb{N}$, and the corresponding embeddings are continuous.

Remark 2.1.3. An embedding of two topological vector spaces $A \subset B$ is called *continuous* if the topology of B induces on A a topology which is weaker than the topology of A, i.e. the intersections of the open sets of B with A are open in A. If A and B are normed spaces, then the embedding $A \subset B$ is continuous if and only if there exists a constant $C > 0$ such that for any $w \in A$ one has $\|w\|_B \leq C\|w\|_A$. We recall also that the embedding $A \subset B$ is called *compact* if any bounded in A set is relatively compact in B.

Remark 2.1.4. The third embedding in (2.1.2) is understood in a special sense. For any $u \in L_p(\Omega)$ one can consider the linear functional $\varphi(u) \in W_p^{-r}(\Omega) = \overset{\circ}{W}{}_q^r(\Omega)^*$, $\frac{1}{p} + \frac{1}{q} = 1$, determined by the formula

$$\langle \varphi(u), \psi \rangle = \int_\Omega \psi(x) u(x)\, dx, \quad \psi \in \overset{\circ}{W}{}_q^r(\Omega).$$

The functional $\varphi(u)$ and the function u are usually identified. Thus, the forth embedding in (2.1.2) and the third one (for $p = 2$) imply, in particular,

$$\langle u, \psi \rangle = (u, \psi), \quad u \in L_2(\Omega), \quad \psi \in \mathcal{D}(\Omega).$$

More information concerning interrelations of the Sobolev and Sobolev–Slobodetskii spaces for various parameters p, r and s is given by the Sobolev embedding theorems and Rellich–Kondrashov compactness theorems (see e.g. [1, 11, 43, 59]). Here we recall the embeddings which will be used in this book.

Theorem 2.1.1. a) *Let Ω be an arbitrary domain in $\mathbb{R}^n$, and let $1 < p_1 \le p_2 < \infty$, $r_1 \in \mathbb{N} \cup \{0\}$, $r_2 \in \mathbb{Z}$. If*

$$\frac{n}{p_1} - r_1 \le \frac{n}{p_2} - r_2, \tag{2.1.3}$$

then

$$\overset{\circ}{W}{}^{r_1}_{p_1}(\Omega) \subset W^{r_2}_{p_2}(\Omega), \tag{2.1.4}$$

and the embedding is continuous. If Ω is sufficiently regular (see Remark 2.1.5), then (2.1.3) implies

$$W^{r_1}_{p_1}(\Omega) \subset W^{r_2}_{p_2}(\Omega) \tag{2.1.5}$$

with continuous embedding.

b) *Let $s \in \mathbb{R}$, $r \in \mathbb{Z}$, $s \ge r \ge 0$, $2 \le p < \infty$. If*

$$\frac{n}{2} - s \le \frac{n}{p} - r, \tag{2.1.6}$$

then

$$H^s_0(\Omega) \subset W^r_p(\Omega), \tag{2.1.7}$$

and the embedding is continuous. If, in addition, Ω is sufficiently regular, then

$$H^s(\Omega) \subset W^r_p(\Omega) \tag{2.1.8}$$

with continuous embedding.

Remark 2.1.5. We call a domain $\Omega \subset \mathbb{R}^n$ *sufficiently regular* if its boundary $\partial\Omega$ is a C^k-smooth manifold for some $k \in \mathbb{N}$ and if Ω is locally located on one side of $\partial\Omega$. In particular, the domain $\Omega = \mathbb{R}^n$ with empty boundary is considered to be sufficiently regular. Of course, embeddings (2.1.5) and (2.1.8) are valid for wider classes of domains (see the references above), but we shall not use such embeddings in this book. Moreover, most of the results will be based on the embeddings like (2.1.4) which do not require any regularity of Ω.

Theorem 2.1.2. *Assume that under the conditions of Theorem 2.1.1 the domain Ω is bounded and inequalities (2.1.3) and (2.1.6) are strict. Then embeddings (2.1.4), (2.1.5), (2.1.7) and (2.1.8) are compact.*

Theorem 2.1.3. *Let Ω be an arbitrary domain in $\mathbb{R}^n$.*

a) *Let $1 < p < \infty$, $r \in \mathbb{N}$. If*
$$\frac{n}{p} < r, \tag{2.1.9}$$
then
$$\overset{\circ}{W}{}_p^r(\Omega) \subset L_\infty(\Omega), \tag{2.1.10}$$
and any function from $\overset{\circ}{W}{}_p^r(\Omega)$ is almost everywhere equal to a continuous on Ω function. If Ω is sufficiently regular, then
$$W_p^r(\Omega) \subset L_\infty(\Omega), \tag{2.1.11}$$
and any function from $W_p^r(\Omega)$ is almost everywhere equal to a continuous on Ω function.

b) *If*
$$s \in \mathbb{R}, \quad n < 2s, \tag{2.1.12}$$
then
$$H_0^s(\Omega) \subset L_\infty(\Omega), \tag{2.1.13}$$
and any function from $H_0^s(\Omega)$ is almost everywhere equal to a continuous on Ω function. If Ω is sufficiently regular, then
$$H^s(\Omega) \subset L_\infty(\Omega), \tag{2.1.14}$$
and any function from $H^s(\Omega)$ is almost everywhere equal to a continuous on Ω function.

c) *Embeddings (2.1.10), (2.1.11), (2.1.13), (2.1.14) are continuous. If, in addition, Ω is bounded, then they are also compact.*

In particular, for $n = 2, 3$ one has
$$H_0^1(\Omega) \subset L_4(\Omega), \tag{2.1.15}$$
$$H_0^s(\Omega) \subset L_\infty(\Omega), \quad s > \frac{3}{2}, \tag{2.1.16}$$
and for sufficiently regular Ω:
$$H^1(\Omega) \subset L_4(\Omega), \tag{2.1.17}$$
$$H^s(\Omega) \subset L_\infty(\Omega), \quad s > \frac{3}{2}. \tag{2.1.18}$$
If Ω is bounded, embeddings (2.1.15) – (2.1.18) are compact.

Corollary 2.1.1. *If Ω is sufficiently regular, $n = 2, 3$, then the following inequalities are valid:*

$$\|uv\| \le C\|u\|_2\|v\|, \quad u \in W_2^2(\Omega), \quad v \in L_2(\Omega), \tag{2.1.19}$$

$$\|uv\| \le C\|u\|_1\|v\|_1, \quad u, v \in W_2^1(\Omega), \tag{2.1.20}$$

$$\|uv\|_1 \le C\|u\|_1\|v\|_2, \quad u \in W_2^1(\Omega), \quad v \in W_2^2(\Omega), \tag{2.1.21}$$

$$\|uv\|_2 \le C\|u\|_2\|v\|_2, \quad u, v \in W_2^2(\Omega). \tag{2.1.22}$$

Remark 2.1.6. Hereafter in this section C stands for various constants independent of u, v.

Proof. To obtain (2.1.19), we apply Hölder's inequality (2.1.1) with $p = p_2 = 2$, $p_1 = \infty$ and use (2.1.18) with $s = 2$:

$$\|uv\| \le \|u\|_{L_\infty(\Omega)}\|v\| \le C\|u\|_2\|v\|.$$

To get (2.1.20), we use (2.1.1) with $p_1 = p_2 = 4$, $p = 2$ and embedding (2.1.17):

$$\|uv\| \le \|u\|_{L_4(\Omega)}\|v\|_{L_4(\Omega)} \le C\|u\|_1\|v\|_1.$$

Now, we have

$$\|uv\|_1 = \big(\|uv\|^2 + \sum_{i=1}^n \|\frac{\partial(uv)}{\partial x_i}\|^2\big)^{1/2} \le \big(\|uv\|^2 + \sum_{i=1}^n \big(\|\frac{\partial u}{\partial x_i}v\| + \|u\frac{\partial v}{\partial x_i}\|\big)^2\big)^{1/2}.$$

Due to (2.1.19) and (2.1.20), this does not exceed

$$C\Big(\|u\|_1^2\|v\|_1^2 + \sum_{i=1}^n \big(\|\frac{\partial u}{\partial x_i}\|\|v\|_2 + \|u\|_1\|\frac{\partial v}{\partial x_i}\|_1\big)^2\Big)^{1/2} \le C\|u\|_1\|v\|_2.$$

Finally, applying (2.1.19) – (2.1.21), we have

$$\begin{aligned}
\|uv\|_2 &= \Big(\|uv\|_1^2 + \sum_{i,j=1}^n \|\frac{\partial^2(uv)}{\partial x_i \partial x_j}\|^2\Big)^{1/2} \\
&\le \Big(\|uv\|_1^2 + \sum_{i,j=1}^n \big(\|\frac{\partial^2 u}{\partial x_i \partial x_j}v\| + \|\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\| + \|\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\| + \|u\frac{\partial^2 v}{\partial x_i \partial x_j}\|\big)^2\Big)^{1/2} \\
&\le \Big(\|u\|_1^2\|v\|_2^2 + \sum_{i,j=1}^n \big(\|\frac{\partial^2 u}{\partial x_i \partial x_j}\|\|v\|_2 + \|\frac{\partial u}{\partial x_i}\|_1\|\frac{\partial v}{\partial x_j}\|_1 \\
&\quad + \|\frac{\partial u}{\partial x_j}\|_1\|\frac{\partial v}{\partial x_i}\|_1 + \|u\|_2\|\frac{\partial^2 v}{\partial x_i \partial x_j}\|\big)^2\Big)^{1/2} \\
&\le C\|u\|_2\|v\|_2,
\end{aligned}$$

and (2.1.22) is proven. □

Let Z denote any of the classes introduced above (L_p, H^s etc.) We shall use the notations $Z(\Omega)^n$ and $Z(\Omega, \mathbb{R}^n)$ for the Cartesian product of n spaces $Z(\Omega)$. The norm in this space is defined as

$$\|u\|_{Z(\Omega)^n} = \Big(\sum_{i=1}^{n} \|u_i\|^2_{Z(\Omega)}\Big)^{1/2},$$

where

$$u = (u_1, u_2, \dots, u_n);$$

the scalar product (if applicable) may be given as

$$(u, v)_{Z(\Omega)^n} = \sum_{i=1}^{n} (u_i, v_i)_{Z(\Omega)};$$

the "bra–ket" product (for $Z = \overset{\circ}{W}{}^r_q(\Omega), \mathcal{D}$) is

$$\langle u, v\rangle_{Z^*(\Omega)^n \times Z(\Omega)^n} = \sum_{i=1}^{n} \langle u_i, v_i\rangle_{Z^*(\Omega)\times Z(\Omega)}.$$

A useful (but simple) inequality is

$$\|u\|_{Z(\Omega)^n} \leq \sum_{i=1}^{n} \|u_i\|_{Z(\Omega)}.$$

We recall Ladyzhenskaya's inequalities:

$$\|u\|_{L_4(\Omega)} \leq 2^{1/4} \|u\|^{1/2} \|\mathrm{grad}\, u\|^{1/2}_{L_2(\Omega)^n}, \quad n = 2, \tag{2.1.23}$$

$$\|u\|_{L_4(\Omega)} \leq 2^{1/2} \|u\|^{1/4} \|\mathrm{grad}\, u\|^{3/4}_{L_2(\Omega)^n}, \quad n = 3, \tag{2.1.24}$$

for any domain $\Omega \subset \mathbb{R}^n$ and $u \in H^1_0(\Omega)$. Here

$$(\mathrm{grad}\, u)_j = \frac{\partial u}{\partial x_j}.$$

See e.g. [61], Lemmas III.3.3 and III.3.5, for the proofs of these inequalities. Note that (2.1.23) and (2.1.24) imply

$$\|u\|_{L_4(\Omega)} \leq 2^{1/2} \|u\|^{1/4} \|u\|^{3/4}_1, \quad n = 2, 3. \tag{2.1.25}$$

Let us mention also Friedrichs' inequality. For any bounded domain Ω:

$$\|v\| \leq K_0(\Omega) \|\mathrm{grad}\, v\|_{L_2(\Omega)^n}, \quad v \in H^1_0(\Omega), \tag{2.1.26}$$

where the constant K_0 is independent of v (see e.g. [20]).

2.1.2 The spaces used in hydrodynamics

In this subsection we are going to introduce the function spaces suitable for description of velocities and stress tensors.

Let

$$\mathcal{V} = \{u \in C_0^\infty(\Omega, \mathbb{R}^n),\ \operatorname{div} u = 0\}.$$

The symbols $H = H(\Omega)$, $V = V(\Omega)$, $V_\delta = V_\delta(\Omega)$ ($\delta \in (0,1]$) denote the closures of $\mathcal{V}$ in $L_2(\Omega, \mathbb{R}^n)$, $H^1(\Omega, \mathbb{R}^n)$, $H^\delta(\Omega, \mathbb{R}^n)$, respectively.

The space V is especially important in hydrodynamics since the assertion $v \in V$ expresses conditions (1.1.10) and (1.1.15).

As a rule, using the Riesz representation theorem, one identifies H with H^*, and, hence, has (see Section 2.2.8)

$$V \subset H \equiv H^* \subset V^*. \tag{2.1.27}$$

Denote by $\mathbb{R}^{n\times n}$ the space of matrices of the order $n \times n$ with the following scalar product: for $A = (A_{ij})$, $B = (B_{ij})$

$$(A, B)_{\mathbb{R}^{n\times n}} = \sum_{i,j=1}^{n} A_{ij} B_{ij}$$

and by $\mathbb{R}_S^{n\times n}$ its subspace of symmetric matrices.

Denote by $\mathbb{R}^{n\times n\times n}$ the space of ordered collections of n matrices of the order $n\times n$ with the following scalar product: for $A = (A_1, \dots, A_n)$, $B = (B_1, \dots, B_n)$

$$(A, B)_{\mathbb{R}^{n\times n\times n}} = \sum_{i=1}^{n} (A_i, B_i)_{\mathbb{R}^{n\times n}}.$$

The symbol ∇u stands for the Jacobi matrix of a vector function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ (cf. (1.1.2)). The symbol $\nabla\tau$ denotes the ordered collection of the Jacobi matrices of the columns of a matrix function $\tau : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n\times n}$.

In this book the Euclidean norms in $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$, $\mathbb{R}_S^{n\times n}$ are denoted as $|\cdot|$.

Let Z denote any of the classes introduced in the previous subsection. By analogy with the case described in the end of that subsection, we use the notations $Z(\Omega, \mathbb{R}^{n\times n})$, $Z(\Omega, \mathbb{R}_S^{n\times n})$, $Z(\Omega, \mathbb{R}^{n\times n\times n})$ for the Cartesian product of the corresponding number of spaces $Z(\Omega)$. The norms, the scalar and "bra-ket" products in these spaces may be defined analogously. The space $H^s(\Omega, \mathbb{R}_S^{n\times n})$ will also be denoted as $H_M^s(\Omega)$.

As in the case of scalar functions, we shall write $\|u\|$ for $\|u\|_{L_2(\Omega,F)}$, $(u, v)_r$ for $(u, v)_{W_2^r(\Omega,F)}$ and $\|u\|_r$ for $\|u\|_{W_2^r(\Omega,F)}$, (u, v) for $\int_\Omega (u(x), v(x))_F\, dx$. Here F stands for any of the spaces $\mathbb{R}^n, \mathbb{R}^{n\times n}, \mathbb{R}_S^{n\times n}, \mathbb{R}^{n\times n\times n}$.

Sometimes (mostly in Chapter 6) we shall write simply L_p, H^s, W_p^r etc. instead of $L_p(\Omega, F)$, $H^s(\Omega, F)$, $W_p^r(\Omega, F)$ when it is clear from the context which domain Ω and space F are used.

It is clear that for these Cartesian product spaces one has embedding theorems similar to the ones described in Section 2.1.1.

Denote by $Y(\Omega)$ the completion of $\mathcal{V}$ with respect to the Euclidean norm

$$\|u\|_{Y(\Omega)} = \|\nabla u\|,$$

which corresponds to the scalar product

$$(u, v)_{Y(\Omega)} = (\nabla u, \nabla v).$$

If Ω is bounded, (2.1.26) implies

$$\|u\|_1^2 \leq (1 + K_0^2(\Omega))\|\nabla u\|^2, u \in V, \tag{2.1.28}$$

so $Y(\Omega)$ coincides with V (up to equivalent norm).

2.2 Spaces of vector functions

2.2.1 Preliminaries

Let E be a Banach space and let $J = (\alpha, \beta)$ be an interval of the real axis $\mathbb{R}$, $-\infty \leq \alpha < \beta \leq +\infty$.

A function $u : J \to E$ is called *simple* if there exist a finite number of mutually disjoint Lebesgue measurable subsets $B_i \subset J$, $i = 1, 2, \ldots, m$, such that the function u is identically a constant x_i in each B_i and vanishes in $J \setminus \bigcup\limits_{i=1}^{m} B_i$.

The *Bochner integral* of a simple function u is defined by the formula

$$\int_\alpha^\beta u(t)\, dt = \sum_{i=1}^{m} \operatorname{mes}(B_i) x_i .$$

A function $u : J \to E$ is called *Bochner measurable* if there exists a sequence of simple functions $\{u_k\}$ such that for almost all $t \in J$:

$$u_k(t) \xrightarrow[k\to\infty]{} u(t) \ \text{ in } \ E.$$

If the condition

$$\lim_{k\to\infty} \int_\alpha^\beta \|u(t) - u_k(t)\|_E\, dt = 0$$

holds for this sequence, then the function u is called *Bochner integrable*. The Bochner integral of such a function is

$$\int_\alpha^\beta u(t)\, dt = \lim_{k\to\infty} \int_\alpha^\beta u_k(t)\, dt.$$

As well as in the case of scalar functions, if two (Bochner) measurable functions differ on a null set, they are considered to be equal.

Denote by $L_p(\alpha, \beta; E)$, $1 \le p \le \infty$, the sets of such measurable functions $u : J \to E$ that

$$\|u\|_{L_p(\alpha,\beta;E)} = \left(\int_\alpha^\beta \|u(t)\|_E^p \, dt \right)^{\frac{1}{p}} < +\infty \quad (1 \le p < \infty);$$

or

$$\|u\|_{L_p(\alpha,\beta;E)} = \operatorname*{ess\,sup}_{t \in J} \|u(t)\|_E < +\infty \quad (p = \infty).$$

They are Banach spaces with the indicated norms. They are separable provided $p < \infty$ and E is separable, and are reflexive for $1 < p < \infty$ and reflexive E. We recall also that

$$L_p(\alpha, \beta; E)^* = L_{\frac{1}{1-1/p}}(\alpha, \beta; E^*), \quad 1 < p < \infty,$$
$$L_1(\alpha, \beta; E)^* = L_\infty(\alpha, \beta; E^*).$$

It is easy to see that a measurable function $u(t)$ belongs to the space $L_p(\alpha, \beta; E)$ if and only if the scalar function $\|u(t)\|_E$ belongs to $L_p(\alpha, \beta) = L_p(\alpha, \beta; \mathbb{R})$, whereas

$$\|u\|_{L_p(\alpha,\beta;E)} = \| \|u(t)\|_E \|_{L_p(\alpha,\beta)}.$$

A function $u : J \to E$ is Bochner integrable if and only if $u \in L_1(\alpha, \beta; E)$.

Remark 2.2.1. In this chapter, when dealing with bounded intervals J, for convenience we use the interval $J = (0, T)$, $0 < T < \infty$. However, each definition or statement using $J = (0, T)$ is good for any J with $\alpha, \beta \neq \pm\infty$ as well.

Lemma 2.2.1. a) *Let $1 \le p \le q \le \infty$, and $u \in L_q(0, T; E)$. Then $u \in L_p(0, T; E)$, and*

$$\|u\|_{L_p(0,T;E)} \le T^{\frac{1}{p} - \frac{1}{q}} \|u\|_{L_q(0,T;E)}.$$

b) *Let*

$$p, p_1, p_2, q, q_1, q_2 \ge 1, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},$$
$$u \in L_{p_1}(\alpha, \beta; L_{q_1}(\Omega)), \quad v \in L_{p_2}(\alpha, \beta; L_{q_2}(\Omega)).$$

Then the pointwise product uv,

$$uv(t)(x) = u(t)(x)v(t)(x),$$

belongs to the space $L_p(\alpha, \beta; L_q(\Omega))$, and

$$\|uv\|_{L_p(\alpha,\beta;L_q(\Omega))} \le \|u\|_{L_{p_1}(\alpha,\beta;L_{q_1}(\Omega))} \|v\|_{L_{p_2}(\alpha,\beta;L_{q_2}(\Omega))}. \tag{2.2.1}$$

Proof. a) Hölder's inequality (2.1.1) with $\Omega = (0,T)$, $\psi_1 = 1$, $\psi_2 = \|u\|_E$, $\frac{1}{p_1} = \frac{1}{p} - \frac{1}{q}$, $p_2 = q$ implies

$$\| \|u\|_E \|_{L_p(0,T)} \le \|1\|_{L_{p_1}(0,T)} \| \|u\|_E \|_{L_q(0,T)},$$

so

$$\|u\|_{L_p(0,T;E)} \le \|1\|_{L_{p_1}(0,T)} \|u\|_{L_q(0,T;E)}.$$

It remains to observe that

$$\|1\|_{L_{p_1}(0,T)} = \Big(\int_0^T dt \Big)^{\frac{1}{p_1}} = T^{\frac{1}{p_1}} = T^{\frac{1}{p} - \frac{1}{q}}.$$

b) It suffices to apply Hölder's inequality (2.1.1) twice:

$$\begin{aligned} \|uv\|_{L_p(\alpha,\beta;L_q(\Omega))} &= \| \|uv\|_{L_q(\Omega)} \|_{L_p(\alpha,\beta)} \\ &\le \| \|u\|_{L_{q_1}(\Omega)} \|v\|_{L_{q_2}(\Omega)} \|_{L_p(\alpha,\beta)} \\ &\le \| \|u\|_{L_{q_1}(\Omega)} \|_{L_{p_1}(\alpha,\beta)} \| \|v\|_{L_{q_2}(\Omega)} \|_{L_{p_2}(\alpha,\beta)} \\ &= \|u\|_{L_{p_1}(\alpha,\beta;L_{q_1}(\Omega))} \|v\|_{L_{p_2}(\alpha,\beta;L_{q_2}(\Omega))}. \end{aligned}$$ □

Lemma 2.2.1, a), implies that for $1 \le p \le q \le \infty$ the space $L_q(0,T;E)$ is continuously embedded into $L_p(0,T;E)$.

Denote by $L_{p,\mathrm{loc}}(\alpha,\beta;E)$ the set of measurable functions $u : J \to E$ which belong to $L_p(t_1,t_2;E)$ for any $t_1,t_2 \in \mathbb{R}, \alpha < t_1 < t_2 < \beta$.

Denote by $C_w(\overline{J};E)$ the set of the functions $u : \overline{J} \to E$ which are *weakly continuous*, i.e. for each $g \in E^*$ the function $\langle g, u(\cdot)\rangle : \overline{J} \to \mathbb{R}$ is continuous. Here $\overline{J}$ is the closure of J.

Denote by $C(\overline{J};E)$ the set of continuous functions $u : \overline{J} \to E$.

Note that $C([0,T];E)$ is a Banach space for the norm

$$\|u\|_{C([0,T],E)} = \max_{t\in[0,T]} \|u(t)\|_E.$$

It is easy to see that the space $C([0,T];E)$ is continuously embedded into $L_p(0,T;E)$. At $1 \le p < \infty$ the embedding is dense; at $p = \infty$ for any $u \in C([0,T];E)$ one has

$$\|u\|_{C([0,T];E)} = \|u\|_{L_\infty(0,T;E)}.$$

Remark 2.2.2. Let i_p stand for the natural embedding operator from $C([0,T];E)$ into $L_p(0,T;E)$. Besides, $C_w([0,T];E) \subset L_p(0,T;E)$ (see [20], Theorem 1.9). Let j_p be the natural embedding. Hereafter we shall often identify $C([0,T];E)$ and $i_p(C([0,T];E))$, as well as $C_w([0,T];E)$ and $j_p(C_w([0,T];E))$.

Note also that $C([0,+\infty);E)$ is a Fréchet space for the pre-norm

$$\|v\|_{C([0,+\infty);E)} = \sum_{i=0}^{+\infty} 2^{-i} \frac{\|v\|_{C([0,i];E)}}{1+\|v\|_{C([0,i];E)}},$$

and $C((-\infty,+\infty);E)$ is a Fréchet space for the pre-norm

$$\|v\|_{C((-\infty,+\infty);E)} = \sum_{i=0}^{+\infty} 2^{-i} \frac{\|v\|_{C([-i,i];E)}}{1+\|v\|_{C([-i,i];E)}}.$$

For more details on Bochner integrable and measurable functions see, for example, [84].

Denote by $\mathcal{D}'(0,T;E)$ the set of linear continuous maps from the space $\mathcal{D}(0,T) = \mathcal{D}((0,T))$ into the space E considered with the weak topology. The topology on $\mathcal{D}'(0,T;E)$ can be set as follows: a sequence $u_k \in \mathcal{D}'(0,T;E)$ converges to $u_0 \in \mathcal{D}'(0,T;E)$ if for any $\psi \in \mathcal{D}(0,T)$ and any linear continuous functional φ on E one has

$$\varphi(u_k(\psi)) \to \varphi(u_0(\psi)).$$

The elements of $\mathcal{D}'(0,T;E)$ are called distributions on $(0,T)$ with values in E.

Definition 2.2.1. The *generalized (distributional) derivative* of a distribution $u \in \mathcal{D}'(0,T;E)$ is the distribution determined by the formula

$$u'(\psi) = -u(\psi'), \quad \forall\, \psi \in \mathcal{D}(0,T).$$

The operator of differentiation is a linear continuous operator on $\mathcal{D}'(0,T;E)$.

As in the scalar case (Remark 2.1.4), it turns out that every element $u \in L_{1,\mathrm{loc}}(0,T;E)$ can be identified with a distribution from $\mathcal{D}'(0,T;E)$ (i.e. with a weakly continuous linear map from $\mathcal{D}(0,T)$ into E) according to the following formula:

$$u(\psi) = \int_0^T \psi(t)u(t)\,dt, \quad \psi \in \mathcal{D}(0,T).$$

In this sense it is possible to consider $L_{1,\mathrm{loc}}(0,T;E) \subset \mathcal{D}'(0,T;E)$.

Lemma 2.2.2. *Let $u \in \mathcal{D}'(0,T;E)$ and $u' = 0$. Then there exists an element $b \in E$ such that $u \equiv b$, i.e. for any $\psi \in \mathcal{D}(0,T)$:*

$$u(\psi) = b\int_0^T \psi(t)\,dt.$$

Proof. Fix a function $\varphi \in \mathcal{D}(0,T)$ such that $\int_0^T \varphi(t)\,dt = 1$. Let $b = u(\varphi)$. Take an arbitrary function $\psi \in \mathcal{D}(0,T)$. The function $g(s) = \psi(s) - \varphi(s)\int_0^T \psi(t)\,dt$ belongs to $\mathcal{D}(0,T)$. Then the function $\eta(\xi) = \int_0^\xi g(s)\,ds$ also belongs to $\mathcal{D}(0,T)$. Really, it is smooth and vanishes at small ξ. For ξ close to T, we have:

$$\begin{aligned}\eta(\xi) &= \int_0^\xi \psi(s)\,ds - \int_0^\xi \varphi(s)\int_0^T \psi(t)\,dt\,ds\\ &= \int_0^T \psi(s)\,ds - \int_0^T \varphi(s)\int_0^T \psi(t)\,dt\,ds\\ &= \int_0^T \psi(s)\,ds - \int_0^T \psi(t)\,dt = 0.\end{aligned}$$

Now we have

$$\begin{aligned}u(\psi) - b\int_0^T \psi(t)\,dt &= u(\psi) - u(\varphi)\int_0^T \psi(t)\,dt\\ &= u\Big(\psi - \varphi\int_0^T \psi(t)\,dt\Big) = u(\eta') = -u'(\eta) = 0. \qquad \square\end{aligned}$$

Lemma 2.2.3. *Let $u \in \mathcal{D}'(0,T;E)$ and $u' \in L_p(0,T;E)$, $1 \le p \le \infty$. Then $u \in C([0,T];E)$, and the following representation takes place:*

$$u(t) = u(0) + \int_0^t u'(s)\,ds. \tag{2.2.2}$$

Proof. Note first that $u' \in L_p(0,T;E) \subset L_1(0,T;E)$. Since $C([0,T];E)$ is dense in $L_1(0,T;E)$, there exists a sequence $g_n \in C([0,T];E)$ such that $g_n \to u'$ in $L_1(0,T;E)$. By the Newton – Leibnitz formula we have for an arbitrary $\psi \in \mathcal{D}(0,T)$:

$$\int_0^T \left(\int_0^t g_n(s)\,ds \cdot \psi(t)\right)' dt = \left(\int_0^t g_n(s)\,ds \cdot \psi(t)\right)\Big|_0^T = 0.$$

But

$$\left(\int_0^t g_n(s)\,ds \cdot \psi(t)\right)' = g_n(t)\psi(t) + \int_0^t g_n(s)\,ds \cdot \psi'(t).$$

Therefore

$$\int_0^T g_n(t)\psi(t)\,dt + \int_0^T \int_0^t g_n(s)\,ds \cdot \psi'(t)\,dt = 0.$$

Passing to the limit as $n \to \infty$, we obtain

$$\int_0^T u'(t)\psi(t)\,dt + \int_0^T \int_0^t u'(s)\,ds \cdot \psi'(t)\,dt = 0.$$

Using the definition of generalized derivative, we rewrite this as

$$u'(\psi) - \left(\int_0^t u'(s)\,ds\right)'(\psi) = 0.$$

Since the function $\psi \in \mathcal{D}(0,T)$ was taken arbitrary, this yields

$$\left(u - \int_0^t u'(s)\,ds\right)' = 0.$$

By Lemma 2.2.2 there exists b such that $u - \int_0^t u'(s)\,ds = b$. Thus,

$$u = b + \int_0^t u'(s)\,ds.$$

Hence, $u \in C([0,T];E)$. It remains to observe that at $t = 0$ the left-hand side of the obtained equality is $u(0)$, and the right-hand one is b, therefore this equality implies the statement of the lemma. □

In conclusion, let us define some other spaces of vector functions.

Denote by $C^r(0,T;E), r \in \mathbb{N}$, the space of the functions $u \in C(0,T;E)$ which have r continuous derivatives with respect to t. The norm in this space is

$$\|u\|_{C^r(0,T;E)} = \sum_{m=0}^{r} \|u^{(m)}(t)\|_{C(0,T;E)}.$$

Denote by $W_p^r(0,T;E),\ 1 \le p \le \infty, r \in \mathbb{N}$ or $r = 0$, the Sobolev space of the functions $u \in L_p(0,T;E)$ for which the norm

$$\|u\|_{W_p^r(0,T;E)} = \sum_{m=0}^{r} \|u^{(m)}(t)\|_{L_p(0,T;E)}$$

is finite. These spaces are Banach spaces. They are separable provided $p < \infty$ and E is separable, and are reflexive for $1 < p < \infty$ and reflexive E.

Corollary 2.2.1. *One has*

$$W_1^1(0,T;E) \subset C([0,T];E),$$

and the embedding is continuous.

Proof. The inclusion follows immediately from Lemma 2.2.3. The continuity is due to the inequality

$$\begin{aligned}
\max_{t\in[0,T]} \|u(t)\|_E &\le \frac{1}{T}\int_0^T \|u(s)\|_E\,ds + \max_{t,s\in[0,T]} \|u(t)-u(s)\|_E \\
&\le \frac{1}{T}\int_0^T \|u(s)\|_E\,ds + \int_0^T \|u'(s)\|_E\,ds \\
&\le C\|u\|_{W_1^1(0,T;E)}.
\end{aligned}$$

□

Denote by $C_0^\infty(0,T;E)$ the set of smooth functions with values in E which have a compact support in $(0,T)$. Denote by $\overset{\circ}{W}{}_p^r(0,T;E)$ the subspace of $W_p^r(0,T;E)$ which is the closure of $C_0^\infty(0,T;E)$.

Denote by $W_p^{-r}(0,T;E)$ the dual space for the space $\overset{\circ}{W}{}_q^r(0,T;E^*)$, $\frac{1}{p}+\frac{1}{q}=1$. Here it is assumed that E is reflexive and $q<\infty$ (we do not need here a definition for the general case, which becomes more complicated).

Every element $u \in W_p^{-r}(0,T;E) = (\overset{\circ}{W}{}_q^r(0,T;E^*))^*$ can be identified with a distribution from $\mathcal{D}'(0,T;E)$ according to the following formula:

$$\langle \xi, u(\psi)\rangle_{E^*\times E} = \langle u, \xi\psi\rangle_{W_p^{-r}(0,T;E)\times \overset{\circ}{W}{}_q^r(0,T;E^*)}, \quad \psi \in \mathcal{D}(0,T), \quad \xi \in E^*.$$

2.2.2 Classical criteria of compactness

A subset K of a Banach space E is called *compact* if one can select a finite subcovering from every open covering of this set; $K \subset E$ is called *relatively compact* if its closure $\overline{K}$ is compact.

Let us recall the well-known Hausdorff and Arzela–Ascoli criteria of relative compactness.

Theorem 2.2.1. *A set $K \subset E$ is relatively compact if and only if for every $\varepsilon > 0$ there exists a finite set $e_i \subset K$, $i = 1,\dots,j$, such that for every $x \in K$ there is an element e_i such that $\|e_i - x\|_E \le \varepsilon$. This set $\{e_i, i = 1,\dots,j\}$ is called ε-net of the set K.*

Corollary 2.2.2. *If K is a uniform limit of relatively compact sets (i.e. for any $\varepsilon > 0$ there exists a relatively compact set K_ε such that for every $x \in K$ there is $y \in K_\varepsilon$ such that $\|x-y\|_E \le \varepsilon$), then K is relatively compact.*

Proof. Fix $\varepsilon > 0$. Choose a relatively compact set $K_{\frac{\varepsilon}{2}}$. By Theorem 2.2.1 it possesses an $\frac{\varepsilon}{2}$-net $\{e_i,\ i=1,\dots,j_\varepsilon\}$. Then $\{e_i,\ i=1,\dots,j_\varepsilon\}$ is an ε-net for K. Really, for any $x \in K$ there is $y \in K_{\frac{\varepsilon}{2}}$ such that $\|x-y\|_E \le \frac{\varepsilon}{2}$, and for this y there exists e_i such that $\|y-e_i\|_E \le \frac{\varepsilon}{2}$. By the triangle inequality $\|x-e_i\|_E \le \varepsilon$. Since ε has been arbitrary, by Theorem 2.2.1 K is a relatively compact set. □

Theorem 2.2.2. *A set $F \subset C([0,T];E)$ is relatively compact if and only if*

$$F(t) = \{f(t) | f \in F\} \text{ is relatively compact in } E \ \forall t \in [0,T], \tag{2.2.3}$$

$$\text{and } F \text{ is equicontinuous}. \tag{2.2.4}$$

The second condition means that for any $\varepsilon > 0$ there is $\delta > 0$ such that for all $t_1, t_2 \in [0,T]$, $|t_1 - t_2| \le \delta$, and every $f \in F$ one has $\|f(t_2) - f(t_1)\|_E \le \varepsilon$.

2.2.3 Compactness in $L_p(0, T; E)$

For a function $f : [0, T] \to E$ and a number $h > 0$ put $(\tau_h f)(t) = f(t + h)$. Then we have a function $\tau_h f : [-h, T - h] \to E$.

Theorem 2.2.3 ([54]). *Let E be a Banach space. A set $F \subset L_p(0, T; E)$, $1 \le p < \infty$, is relatively compact if and only if*

$$\text{the set } \left\{ \int_{t_1}^{t_2} f(t)\,dt \mid f \in F \right\} \text{ is relatively compact in } E \tag{2.2.5}$$

for every $t_1, t_2 \in (0, T)$, $t_1 < t_2$, and

$$\|\tau_h f - f\|_{L_p(0,T-h;E)} \xrightarrow[h\to 0]{} 0 \text{ uniformly with respect to } f \in F, \tag{2.2.6}$$

i.e. for any $\varepsilon > 0$ there is $\delta > 0$ such that for every $f \in F$ and $h < \delta$ one has $\|\tau_h f - f\|_{L_p(0,T-h;E)} \le \varepsilon$.

Proof. The first stage. Let F be a relatively compact subset of $L_p(0, T; E)$.

Let us show that the linear map $l : f \mapsto \int_{t_1}^{t_2} f(t)\,dt$, $t_1, t_2 \in (0, T)$, $t_1 < t_2$, from $L_p(0, T; E)$ into E, is bounded. Really, using Lemma 2.2.1, a), we get

$$\left\| \int_{t_1}^{t_2} f(t)\,dt \right\|_E \le \int_{t_1}^{t_2} \|f(t)\|_E\,dt \le \|f\|_{L_1(0,T;E)} \le T^{1-\frac{1}{p}} \|f\|_{L_p(0,T;E)}.$$

Therefore the map l is continuous, so it transforms relatively compact sets to relatively compact ones. Thus, (2.2.5) is fulfilled.

Fix $\varepsilon > 0$. By Theorem 2.2.1 there is a finite $\frac{\varepsilon}{5}$-net $\{f_i, i = 1, \dots, j\}$ in the space $L_p(0, T; E)$ for the set F. Since $C([0, T]; E)$ is dense in $L_p(0, T; E)$, there exist $e_i \in C([0, T]; E)$, $\|e_i - f_i\|_{L_p(0,T;E)} \le \frac{\varepsilon}{5}$, $i = 1, \dots, j$. For continuous functions e_i we have $\|\tau_h e_i - e_i\|_{C([0,T-h];E)} \xrightarrow[h\to 0]{} 0$. Therefore $\|\tau_h e_i - e_i\|_{L_p(0,T-h;E)} \xrightarrow[h\to 0]{} 0$, and for δ small enough:

$$\|\tau_h e_i - e_i\|_{L_p(0,T-h;E)} \le \frac{\varepsilon}{5} \quad \text{for} \quad h < \delta.$$

For any element $f \in F$ there is an element f_i from the $\frac{\varepsilon}{5}$-net such that $\|f - f_i\|_{L_p(0,T;E)} \le \frac{\varepsilon}{5}$. By the triangle inequality:

$$\begin{aligned}
&\|\tau_h f - f\|_{L_p(0,T-h;E)} \\
&\le \|\tau_h(f - f_i)\|_{L_p(0,T-h;E)} + \|\tau_h(f_i - e_i)\|_{L_p(0,T-h;E)} \\
&\qquad + \|\tau_h e_i - e_i\|_{L_p(0,T-h;E)} + \|e_i - f_i\|_{L_p(0,T-h;E)} + \|f_i - f\|_{L_p(0,T-h;E)} \\
&= \|f - f_i\|_{L_p(h,T;E)} + \|f_i - e_i\|_{L_p(h,T;E)} + \|\tau_h e_i - e_i\|_{L_p(0,T-h;E)} \\
&\qquad + \|e_i - f_i\|_{L_p(0,T-h;E)} + \|f_i - f\|_{L_p(0,T-h;E)} \\
&\le 5 \cdot \frac{\varepsilon}{5} = \varepsilon.
\end{aligned}$$

We have proved that condition (2.2.6) also holds.

The second stage. Assume, on the contrary, that conditions (2.2.5) and (2.2.6) hold. Put for $f \in F$ and $0 < a < T$:

$$(M_a f)(t) = \frac{1}{a} \int_t^{t+a} f(s)\, ds. \tag{2.2.7}$$

We have for $t_1, t_2 \in [0, T - a],\ \ t_1 < t_2$:

$$\begin{aligned}
\|(M_a f)(t_2) - (M_a f)(t_1)\|_E &= \left\| \frac{1}{a} \left(\int_{t_2}^{t_2+a} f(s)\, ds - \int_{t_1}^{t_1+a} f(s)\, ds \right) \right\|_E \\
&= \frac{1}{a} \left\| \int_{t_1}^{t_1+a} (\tau_{t_2-t_1} f - f)(s)\, ds \right\|_E \le \frac{1}{a} \left\| \tau_{t_2-t_1} f - f \right\|_{L_1(0,T-(t_2-t_1);E)} \\
&\le \frac{T^{1-1/p}}{a} \left\| \tau_{t_2-t_1} f - f \right\|_{L_p(0,T-(t_2-t_1);E)}.
\end{aligned}$$

Condition (2.2.6) yields that the right-hand side tends to zero uniformly with respect to f as $|t_2 - t_1| \to 0$. Hence, the set $M_a F = \{M_a f | f \in F\}$ is contained in $C([0, T - a]; E)$ and is equicontinuous on $[0, T - a]$.

The condition (2.2.5) with $t_1 = t,\ t_2 = t + a,\ t \in (0, T - a)$ gives that the set $(M_a F)(t) = \{\frac{1}{a} \int_t^{t+a} f(s)\, ds | f \in F\}$ is relatively compact in E for all $t \in (0, T - a)$. The set $(M_a F)(0)$ is the uniform limit of the relatively compact sets $(M_a F)(\varepsilon)$ as $\varepsilon \to 0$. In fact, equicontinuity of $M_a F$ implies that for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for any $f \in F$ one has $\|M_a f(\delta) - M_a f(0)\| \le \varepsilon$.

Therefore the set $(M_a F)(0)$ is also relatively compact. Similarly, $(M_a F)(T - a)$ is relatively compact as the uniform limit of the sets $(M_a F)(T - a - \delta)$ as $\delta \to 0$.

We have shown that on the segment $[0, T - a]$ the set $M_a F$ satisfies conditions (2.2.3) and (2.2.4) of Theorem 2.2.2. Hence, $M_a F$ is relatively compact in $C([0, T - a], E)$ and therefore in $L_p(0, T - a; E)$ for any $a \in (0, T)$.

We have for $t \in [0, T - a]$:

$$\begin{aligned}
M_a f(t) - f(t) &= \frac{1}{a} \int_t^{t+a} f(s)\, ds - f(t) \\
&= \frac{1}{a} \int_0^a f(t + h)\, dh - \frac{1}{a} \int_0^a f(t)\, dh = \frac{1}{a} \int_0^a (\tau_h f - f)(t)\, dh.
\end{aligned} \tag{2.2.8}$$

Hence,

$$\|M_a f - f\|_{L_p(0,T-a;E)} \le \sup_{h \in [0,a]} \|\tau_h f - f\|_{L_p(0,T-a;E)}. \tag{2.2.9}$$

Let us show that a set of restrictions $F|_{[0,T_1]} = \{f|_{[0,T_1]} \,\big|\, f \in F\}$ is the uniform limit of the sets $M_a F$ in $L_p(0, T_1; E)$ for every $T_1 \in (0, T)$ as $a \to 0$ (here and below we omit the symbol of restriction). Really, fix $\varepsilon > 0$. For $a < T - T_1$, (2.2.9) yields:

$$\|M_a f - f\|_{L_p(0,T_1;E)} \leq \sup_{h\in[0,a]} \|\tau_h f - f\|_{L_p(0,T-h;E)}. \tag{2.2.10}$$

By condition (2.2.6) there is a such that the right-hand side and, consequently, the left-hand side of (2.2.10) does not exceed ε for any $f \in F$. Thus, F is relatively compact in $L_p(0, T_1; E)$ as the uniform limit of the relatively compact sets $M_a F$.

Let us show that F is relatively compact in $L_p(T - T_1, T; E)$. Consider the set

$$\widetilde{F} = \{\widetilde{f}(t) = f(T-t) | f \in F\}.$$

For $\widetilde{F}$ condition (2.2.5) holds. Let us show that condition (2.2.6) also holds for it. We have for $\widetilde{f} \in \widetilde{F}$ that $\widetilde{f}(t) = f(T-t)$, $f \in F$. Then

$$\|\tau_h\widetilde{f} - \widetilde{f}\|_{L_p(0,T-h;E)} = \left(\int_0^{T-h} \left\|\widetilde{f}(t+h) - \widetilde{f}(t)\right\|_E^p dt\right)^{\frac{1}{p}}$$

$$= \left(\int_0^{T-h} \left\|f(T-t-h) - f(T-t)\right\|_E^p dt\right)^{\frac{1}{p}}$$

$$= \left(-\int_0^{T-h} \left\|f(T-t-h) - f(T-t)\right\|_E^p d(T-t-h)\right)^{\frac{1}{p}}$$

$$= \left(\int_0^{T-h} \left\|f(s) - f(s+h)\right\|_E^p ds\right)^{\frac{1}{p}} = \|f - \tau_h f\|_{L_p(0,T-h;E)}.$$

The last expression tends to zero as $h \to 0$ uniformly with respect to $f \in F$ by condition (2.2.6) for F.

Thus, for $\widetilde{F}$ conditions (2.2.5) and (2.2.6) are fulfilled. Therefore, just as it was shown above for F, $\widetilde{F}$ is relatively compact in $L_p(0, T_1; E)$. It implies that F is relatively compact in $L_p(T - T_1, T; E)$. Taking $T_1 \geq \frac{T}{2}$ we conclude that F is relatively compact in $L_p(0, T; E)$. The theorem is completely proved. □

Remark 2.2.3. For $p = \infty$, the statement of Theorem 2.2.3 as it is formulated above is incorrect. Really, consider the set F consisting of only one function $f(t) = \mathrm{sign}(t - \frac{T}{2})u$, where $u \in E$, $u \neq 0$.

Obviously, it is compact in $L_\infty(0, T; E)$. We have: $\tau_h f(t) - f(t) = u[\mathrm{sign}(t - \frac{T}{2} + h) - \mathrm{sign}(t - \frac{T}{2})]$. Hence, $\tau_h f - f$ is identically equal to $2u$ in the interval $(\frac{T}{2} - h, \frac{T}{2})$. Therefore $\|\tau_h f - f\|_{L_\infty(0,T-h;E)}$ does not converge to zero as $h \to 0$, i.e. condition (2.2.6) is not fulfilled.

Remark 2.2.4. However conditions (2.2.5) and (2.2.6) for $p = \infty$ are close to the conditions of Theorem 2.2.2 on compactness in $C([0,T];E)$. Condition (2.2.6), as we shall see, is equivalent to the equicontinuity (2.2.4), and condition (2.2.5) resembles condition (2.2.3). As the following theorem shows, it is not an accidental coincidence.

Theorem 2.2.4 ([54]). *A set $F \subset L_\infty(0,T;E)$ satisfies conditions* (2.2.5) *and* (2.2.6) *with $p = \infty$ if and only if F is contained in $C([0,T];E)$ and is relatively compact in this space.*

Proof. The first stage. Let F be a relatively compact subset of $C([0,T];E)$. Then F is relatively compact in $L_p(0,T;E)$, and by Theorem 2.2.3 F satisfies condition (2.2.5). By Theorem 2.2.2 F satisfies condition (2.2.4). It remains to observe that (2.2.4) yields (2.2.6) with $p = \infty$. In fact, for $f \in F \subset C([0,T];E)$:

$$\|\tau_h f - f\|_{L_\infty(0,T-h;E)} = \|\tau_h f - f\|_{C([0,T-h];E)} = \max_{t\in[0,T-h]} \|f(x+h) - f(x)\|_E, \tag{2.2.11}$$

which in view of the equicontinuity (2.2.4) tends to zero uniformly with respect to f as $h \to 0$.

The second stage. Assume, on the contrary, that conditions (2.2.5) and (2.2.6) with $p = \infty$ hold. As in the second stage of the proof of Theorem 2.2.3, it is possible to consider the set $M_a F$ and to verify that it is relatively compact in $C([0,T-a];E)$ for every $a \in (0,T)$. Then (2.2.8) yields (2.2.9) and (2.2.10) for $p = \infty$. As in the proof of Theorem 2.2.3, condition (2.2.6) implies that $M_a f \xrightarrow[a\to 0]{} f$ in $L_\infty(0,T_1;E)$, $T_1 \in (0,T)$, uniformly with respect to $f \in F$. Then for any sequence $a_n \to 0$ the sequence $M_{a_n} f$ is fundamental:

$$\|M_{a_n} f - M_{a_m} f\|_{L_\infty(0,T_1;E)} = \|M_{a_n} f - M_{a_m} f\|_{C([0,T_1];E)} \xrightarrow[\max(n,m)\to\infty]{} 0.$$

Since the space $C([0,T_1];E)$ is complete, $M_a f \xrightarrow[a\to 0]{} f$ in $C([0,T_1];E)$ uniformly with respect to $f \in F$, i.e. F is the uniform limit of the relatively compact sets $M_a F$ in $C([0,T_1];E)$. Therefore F is relatively compact in $C([0,T_1];E)$. As well as in the proof of Theorem 2.2.3, having considered the set $\widetilde{F} = \{f(T-t) | f \in F\}$, which satisfies conditions (2.2.5) and (2.2.6) and therefore is relatively compact in $C([0,T_1];E)$, we conclude that F is relatively compact in $C([T-T_1,T];E)$. Taking $T_1 > \frac{T}{2}$ we get that F is relatively compact in $C([0,T];E)$. $\square$

Remark 2.2.5. At the first stage of the proof of Theorem 2.2.4 we have shown that for $F \subset L_\infty(0,T;E)$ the condition of equicontinuity (2.2.4) implies (2.2.6) with $p = \infty$; at the second stage we have got that condition (2.2.6) with $p = \infty$ yields that, for any $f \in F$, $M_a f \xrightarrow[a\to 0]{} f$ in $C([0,T_1];E)$. Hence, $f \in C([0,T_1];E)$ for $f \in F$. But if F satisfies (2.2.6), then $\widetilde{F}$ satisfies (2.2.6), so for $\widetilde{f} \in F$ one has

$M_a \widetilde{f} \xrightarrow[a \to 0]{} \widetilde{f}$ in $C([0, T_1]; E)$, which gives $f \in C([T - T_1, T]; E)$ for all $f \in F$. Taking $T_1 > \frac{T}{2}$, we get $F \subset C([0, T]; E)$. Then (2.2.6) and (2.2.11) yield (2.2.4). Thus, for $F \subset L_\infty(0, T; E)$ conditions (2.2.4) and (2.2.6) are equivalent.

2.2.4 Compactness of sets of vector functions with values in an "intermediate" space

Let $X \subset E \subset Y$ be Banach spaces where the embedding $X \subset E$ is compact, and $E \subset Y$ continuously.

Lemma 2.2.4 ([37]). *For any $\eta > 0$ there exists a natural number N such that for all $u \in X$:*

$$\|u\|_E \leq \eta \|u\|_X + N \|u\|_Y. \tag{2.2.12}$$

Proof. For every natural n, denote by V_n the set $\{u \in E \,\big|\, \|u\|_E - \eta - n\|u\|_Y < 0\}$. Since E is embedded into Y continuously, the function $\|u\|_E - \eta - n\|u\|_Y : E \to \mathbb{R}$ is continuous. Therefore the sets V_n are open. As n increases, the sequence of sets V_n extends and their unit covers the whole space E. Since the embedding $X \subset E$ is compact, the unit sphere S of the space X is compact in E. Therefore from the open covering $\{V_n\}$ of the set S one can select a finite subcovering $\{V_{n_1}, V_{n_2}, \dots, V_{n_j}\}$. Let $N = \max(n_1, n_2, \dots, n_j)$. Then $S \subset V_N$, i.e. for any $u \in S$:

$$\|u\|_E - \eta - N \|u\|_Y < 0. \tag{2.2.13}$$

Now, if $u \neq 0$, the element $\frac{u}{\|u\|_X} \in S$ satisfies estimate (2.2.13), i.e.

$$\left\| \frac{u}{\|u\|_X} \right\|_E - \eta - N \left\| \frac{u}{\|u\|_X} \right\|_Y < 0.$$

Multiplying by $\|u\|_X$, we obtain estimate (2.2.12). If $u = 0$, estimate (2.2.12) is obvious. □

On application of the compactness theorems obtained in this section, difficulties usually arise at the check of condition (2.2.6). The use of the space triples $X \subset E \subset Y$ gives opportunity to weaken this condition in exchange for a strengthening of condition (2.2.5).

Theorem 2.2.5 ([54]). *Let $X \subset E \subset Y$ be Banach spaces where the embedding $X \subset E$ is compact, and $E \subset Y$ continuously. Let $F \subset L_p(0, T; X)$, $1 \leq p \leq \infty$. If*

$$F \text{ is bounded in } L_p(0, T; X), \tag{2.2.14}$$

$$\|\tau_h f - f\|_{L_p(0, T-h; Y)} \xrightarrow[h \to 0]{} 0 \text{ uniformly with respect to } f \in F, \tag{2.2.15}$$

then F is relatively compact in $L_p(0, T; E)$ (at $p = \infty$ the set F is contained in $C([0, T]; E)$ and is relatively compact in this space).

Proof. It suffices to check that conditions (2.2.5) and (2.2.6) hold: in this case the statement of the theorem follows from Theorem 2.2.3 (or Theorem 2.2.4, for $p = \infty$).

As we have already observed, the linear map $f \mapsto \int_{t_1}^{t_2} f(t)\,dt$, $t_1, t_2 \in (0,T)$, $t_1 < t_2$, is continuous from $L_p(0,T;X)$ into X. Therefore, since F is bounded in $L_p(0,T;X)$, the set $\{\int_{t_1}^{t_2} f(t)\,dt \mid f \in F\}$ is bounded in X. Hence, this set is relatively compact in E, i.e. the condition (2.2.5) holds.

Let us show now that the condition (2.2.6) is fulfilled. Fix $\varepsilon > 0$. Since F is bounded in $L_p(0,T;X)$, there is a number $R > 0$ such that $\|f\|_{L_p(0,T;X)} < R$, $f \in F$. Applying Lemma 2.2.4 with $\eta = \frac{\varepsilon}{4R}$, we get that for some N and almost all $t \in (0,T)$:

$$\|\tau_h f(t) - f(t)\|_E \le \frac{\varepsilon}{4R}\|\tau_h f(t) - f(t)\|_X + N\|\tau_h f(t) - f(t)\|_Y.$$

Therefore

$$\|\tau_h f - f\|_{L_p(0,T-h;E)} \le \frac{\varepsilon}{4R}\|\tau_h f - f\|_{L_p(0,T-h;X)} + N\|\tau_h f - f\|_{L_p(0,T-h;Y)}. \tag{2.2.16}$$

By condition (2.2.15) the second term in the right-hand side of (2.2.16) uniformly with respect to f tends to zero as $h \to 0$ and for h small enough does not exceed $\frac{\varepsilon}{2}$. On the other hand,

$$\begin{aligned}\frac{\varepsilon}{4R}\|\tau_h f - f\|_{L_p(0,T-h;X)} &\le \frac{\varepsilon}{4R}\left(\|\tau_h f\|_{L_p(0,T-h;X)} + \|f\|_{L_p(0,T-h;X)}\right)\\ &= \frac{\varepsilon}{4R}\left(\|f\|_{L_p(h,T;X)} + \|f\|_{L_p(0,T-h;X)}\right) \le \frac{\varepsilon}{4R}(R+R) = \frac{\varepsilon}{2}.\end{aligned}$$

Now (2.2.16) implies

$$\|\tau_h f - f\|_{L_p(0,T-h;E)} \le \varepsilon,$$

for all $f \in F$. Thus, condition (2.2.6) also holds. □

2.2.5 The Aubin–Simon theorem

The compactness theorems are frequently applied to establish relative compactness of sets of solutions for differential equations. Here it often happens that it is known that the set of solutions is bounded in a comparatively narrow space $L_p(0,T;X)$, and the set of their derivatives with respect to t is bounded in some "weak sense", for example, in the metric of a space $L_r(0,T;Y)$ wide enough. In this case one may often apply the following theorem.

Theorem 2.2.6. *Let $X \subset E \subset Y$ be Banach spaces where the embedding $X \subset E$ is compact, and $E \subset Y$ continuously. Let $F \subset L_p(0,T;X)$, $1 \le p \le \infty$. Assume* *that for any* $f \in F$ *its generalized derivative* f' *in the space* $\mathcal{D}'(0, T; Y)$ *belongs to* $L_r(0, T; Y),\ 1 \le r \le \infty$. *Let*

$$F \text{ be bounded in } L_p(0, T; X), \tag{2.2.17}$$

$$\{f' | f \in F\} \text{ be bounded in } L_r(0, T; Y). \tag{2.2.18}$$

Then for $p < \infty$ F *is relatively compact in* $L_p(0, T; E)$; *for* $p = \infty$ *and* $r > 1$, F *is relatively compact in* $C([0, T]; E)$.

Remark 2.2.6. The first proof of this statement was given by Aubin [8] for the case when $r > 1,\ 1 < p < \infty$, and the spaces X and Y are reflexive. Aubin's result was generalized by several authors (for example, Dubinskii [19] gave a proof for the case $p = 2,\ r = 1,\ X,\ E,\ Y$ are Hilbert spaces). In the general case the theorem was formulated and proved by Simon [54].

Remark 2.2.7. In the case $p = \infty,\ r = 1$ the statement of Theorem 2.2.6 is incorrect. Really, fix an arbitrary nonzero element $b \in X$ and an arbitrary smooth function $\varphi : \mathbb{R} \to \mathbb{R},\ \varphi \not\equiv 0$, vanishing outside of the interval $(0, T)$. Consider the set

$$G = \{g_n(t) = b\varphi(nt) | n \in \mathbb{N}\}.$$

We have:

$$\begin{aligned}
\|\varphi(nt)\|_{L_\infty(0,T)} &= \operatorname*{ess\,sup}_{t\in(0,T)} |\varphi(nt)| = \operatorname*{ess\,sup}_{s\in(0,nT)} |\varphi(s)| \\
&= \|\varphi\|_{L_\infty(0,nT)} = \|\varphi\|_{L_\infty(0,T)}, \\
\|\varphi(nt)\|_{L_1(0,T)} &= \int_0^T |\varphi(nt)|\, dt = \frac{1}{n}\int_0^{nT} |\varphi(nt)|\, d(nt) \\
&= \frac{1}{n}\|\varphi\|_{L_1(0,nT)} = \frac{1}{n}\|\varphi\|_{L_1(0,T)}, \\
\|(\varphi(nt))'\|_{L_1(0,T)} &= \|n\varphi'(nt)\|_{L_1(0,T)} = n \cdot \frac{1}{n}\|\varphi'\|_{L_1(0,T)} = \|\varphi'\|_{L_1(0,T)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|g_n\|_{L_\infty(0,T;X)} &= \Big\| \|b\varphi(nt)\|_X \Big\|_{L_\infty(0,T)} \\
&= \|b\|_X \|\varphi(nt)\|_{L_\infty(0,T)} = \|b\|_X \|\varphi\|_{L_\infty(0,T)},
\end{aligned}$$

i.e. the set G is bounded in $L_\infty(0, T; X)$;

$$\begin{aligned}
\|g_n\|_{L_1(0,T;E)} &= \Big\| \|b\varphi(nt)\|_E \Big\|_{L_1(0,T)} \\
&= \|b\|_E \|\varphi(nt)\|_{L_1(0,T)} = \frac{1}{n}\|b\|_E \|\varphi\|_{L_1(0,T)},
\end{aligned}$$

i.e.

$$g_n \xrightarrow[n\to\infty]{} 0 \text{ in } L_1(0,T;E); \tag{2.2.19}$$

$$\begin{aligned}\|g_n'\|_{L_1(0,T;Y)} &= \Big\| \|(b\varphi(nt))'\|_Y \Big\|_{L_1(0,T)} \\ &= \|b\|_Y \|(\varphi(nt))'\|_{L_1(0,T)} = \|b\|_Y \|\varphi'\|_{L_1(0,T)},\end{aligned}$$

i.e. the set $\{g_n'\}$ is bounded in $L_1(0,T;Y)$.

Thus, the set G satisfies conditions of Theorem 2.2.6 with $p = \infty$ and $r = 1$. However the set G is not relatively compact either in $C([0,T];E)$ or in $L_\infty(0,T;E)$. Assume the contrary. Then there is a subsequence $\{g_{n_m}\} \subset G$ and an element $g \in L_\infty(0,T;E)$ such that $g_{n_m} \xrightarrow[m\to\infty]{} g$ in $L_\infty(0,T;E)$. It yields $g_{n_m} \xrightarrow[m\to\infty]{} g$ in $L_1(0,T;E)$, and (2.2.19) gives that $g = 0$. So

$$g_{n_m} \xrightarrow[m\to\infty]{} 0 \text{ in } L_\infty(0,T;E).$$

On the other hand,

$$\|g_{n_m}\|_{L_\infty(0,T;E)} = \|b\|_E \|\varphi(nt)\|_{L_\infty(0,T)} = \|b\|_E \|\varphi\|_{L_\infty(0,T)}.$$

Passing to the limit as $m \to \infty$, we get

$$0 = \|b\|_E \|\varphi\|_{L_\infty(0,T)},$$

which contradicts the assumptions $b \neq 0$ and $\varphi \not\equiv 0$.

For the proof of Theorem 2.2.6 we need the following

Lemma 2.2.5. *Let* $u \in L_r(0,T;Y),\ r \geq 1,\ h \in (0,T)$. *Then for the function*

$$J_h u(t) = \int_t^{t+h} u(s)\,ds = \int_0^h u(t+s)\,ds$$

the following estimates take place

$$\|J_h u\|_{L_p(0,T-h;Y)} \leq T^{1-\frac{1}{r}} h^{\frac{1}{p}} \|u\|_{L_r(0,T;Y)}, \quad 1 \leq p < \infty, \tag{2.2.20}$$

$$\|J_h u\|_{L_p(0,T-h;Y)} \leq h^{1-\frac{1}{r}} \|u\|_{L_r(0,T;Y)}, \quad p = \infty. \tag{2.2.21}$$

Proof. Let k be a natural number such that $kh \geq T$. Define the function $\widetilde{u} : [0,kh] \to Y$ by the rule: $\widetilde{u}(t) = u(t)$ for $t \in [0,T],\ \ u(t) = 0$ for $t > T$. We have:

$$\begin{aligned}\|J_h u\|_{L_p(0,T-h;Y)} &\leq \|J_h \widetilde{u}\|_{L_p(0,kh-h;Y)} \\ &= \left(\int_0^{kh-h} \left(\int_0^h \|\widetilde{u}(t+s)\|\,ds \right)^p dt \right)^{\frac{1}{p}} \\ &= \left(\sum_{j=0}^{k-2} \int_0^h \left(\int_0^h \|\widetilde{u}(t+jh+s)\|_Y\,ds \right)^p dt \right)^{\frac{1}{p}}.\end{aligned} \tag{2.2.22}$$

Using the simple scalar inequality

$$\sum_{j=0}^{m} a_j^p \le \left(\sum_{j=0}^{m} a_j\right)^p, \quad a_0, \dots, a_m \ge 0, \quad p \ge 1,$$

we conclude that the right-hand side of (2.2.22) does not exceed

$$\begin{aligned}
&\left(\int_0^h \left(\sum_{j=0}^{k-2} \int_0^h \|\widetilde{u}(t + jh + s)\|_Y \, ds\right)^p dt\right)^{\frac{1}{p}} \\
&= \left(\int_0^h \left(\int_0^{kh-h} \|\widetilde{u}(t + s)\|_Y \, ds\right)^p dt\right)^{\frac{1}{p}} \\
&= \left(\int_0^h \left(\int_t^{kh-h+t} \|\widetilde{u}(s)\|_Y \, ds\right)^p dt\right)^{\frac{1}{p}} \\
&\le \left(\int_0^h \left(\int_0^{kh} \|\widetilde{u}(s)\|_Y \, ds\right)^p dt\right)^{\frac{1}{p}} = \left(h\left(\int_0^T \|u(s)\|_Y \, ds\right)^p\right)^{\frac{1}{p}} \\
&= h^{\frac{1}{p}} \int_0^T \|u(s)\|_Y \, ds = h^{\frac{1}{p}} \|u\|_{L_1(0,T;Y)},
\end{aligned}$$

which does not exceed $h^{\frac{1}{p}} T^{1-\frac{1}{r}} \|u\|_{L_r(0,T;Y)}$ in view of Lemma 2.2.1, a). Estimate (2.2.20) is proved. We have then:

$$\|J_h u\|_{L_\infty(0,T-h;Y)} = \operatorname*{ess\,sup}_{t\in(0,T-h)} \|J_h u(t)\|_Y \le \operatorname*{ess\,sup}_{t\in(0,T-h)} \int_0^h \|u(t + s)\|_Y \, ds.$$

Lemma 2.2.1, a), gives that the right-hand side does not exceed

$$\begin{aligned}
\operatorname*{ess\,sup}_{t\in(0,T-h)} h^{1-\frac{1}{r}} \left(\int_0^h \|u(t + s)\|_Y^r\right)^{\frac{1}{r}} &\le h^{1-\frac{1}{r}} \left(\int_0^T \|u(s)\|_Y^r \, ds\right)^{\frac{1}{r}} \\
&= h^{1-\frac{1}{r}} \|u\|_{L_r(0,T;Y)}.
\end{aligned}$$

□

Proof of Theorem 2.2.6. It suffices to check the conditions of Theorem 2.2.5. The condition (2.2.14) coincides with (2.2.17).

For $f \in F$ we have $f' \in L_r(0, T; Y)$. Then by formula (2.2.2):

$$\begin{aligned}
(\tau_h f - f)(t) &= f(t + h) - f(t) \\
&= f(0) + \int_0^{t+h} f'(s) \, ds - f(0) - \int_0^t f'(s) \, ds = \int_t^{t+h} f'(s) \, ds.
\end{aligned}$$

So by Lemma 2.2.5:

$$\|\tau_h f - f\|_{L_p(0,T-h;Y)} \le T^{1-\frac{1}{r}} h^{\frac{1}{p}} \|f'\|_{L_r(0,T;Y)}, \quad 1 \le p \le \infty,$$
$$\|\tau_h f - f\|_{L_p(0,T-h;Y)} \le h^{1-\frac{1}{r}} \|f'\|_{L_r(0,T;Y)}, \quad p = \infty.$$

The right-hand sides tend to zero as $h \to 0$ uniformly with respect to $f \in F$, therefore (2.2.15) holds. □

2.2.6 Theorem on "partial" compactness

Using the same technique as has been just described, Simon [54] proved the following interesting result.

Theorem 2.2.7. *Let $X \subset E \subset Y$ be Banach spaces where the embedding $X \subset E$ is compact, and $E \subset Y$ continuously. Let $F \subset L_{1,\mathrm{loc}}(0,T;X) \cap L_p(0,T;E)$, $1 < p \le \infty$. Assume that for any $f \in F$ its generalized derivative f' in the space $\mathcal{D}'(0,T;Y)$ belongs to $L_{1,\mathrm{loc}}(0,T;Y)$. Let*

$$F \text{ be bounded in } L_p(0,T;E). \tag{2.2.23}$$

Assume that for any $t_1, t_2 \in (0,T)$, $t_1 < t_2$, the set

$$\{f|_{[t_1,t_2]} \mid f \in F\} \text{ is bounded in } L_1(t_1,t_2;X), \tag{2.2.24}$$

while the set

$$\{f'|_{[t_1,t_2]} \mid f \in F\} \text{ is bounded in } L_1(t_1,t_2;Y). \tag{2.2.25}$$

Then for any $q < p$ F is relatively compact in $L_q(0,T;E)$.

2.2.7 Lemma on weak continuity of essentially bounded functions

The following lemma by J.-L. Lions and E. Magenes illustrates the subtle connection between boundedness and continuity.

Lemma 2.2.6. *Let E and E_0 be Banach spaces, $E \subset E_0$ continuously. Let the space E be reflexive. Then*

$$C_w([0,T];E_0) \cap L_\infty(0,T;E) \subset C_w([0,T];E).$$

Remark 2.2.8. See the proof in [61], Chapter III, Lemma 1.4, as well as [55], p. 232 where it is also shown that without the condition of reflexivity of E there may be no such embedding.

Remark 2.2.9. Lemma 2.2.6 shows that the values of functions from $C_w([0,T];E_0) \cap L_\infty(0,T;E)$ belong to E at every $t \in [0,T]$.

2.2.8 Lemma on differentiability of the quadrate of the norm of a vector function

Assume that we have two Hilbert spaces, $X \subset Y$ with continuous embedding operator $i : X \to Y$, and $i(X)$ is dense in Y. The adjoint operator $i^* : Y^* \to X^*$ is continuous and, since $i(X)$ is dense in Y, one-to-one. Since i is one-to-one, $i^*(Y^*)$ is dense in X^*, and one may identify Y^* with a dense subspace of X^*. Due to the Riesz representation theorem, one may also identify Y with Y^*. We arrive at the chain of inclusions:

$$X \subset Y \equiv Y^* \subset X^*. \tag{2.2.26}$$

Both embeddings here are dense and continuous. Observe that in this situation, for $f \in Y, u \in X$, their scalar product in Y coincides with the value of the functional f from X^* on the element $u \in X$:

$$(f, u)_Y = \langle f, u \rangle. \tag{2.2.27}$$

The following lemma is a particular case of a general interpolation theorem by Lions and Magenes.

Lemma 2.2.7. *If a function u belongs to $L_2(0, T; X)$ and its derivative u' belongs to $L_2(0, T; X^*)$, then $u \in C([0, T]; Y)$, and*

$$\frac{d}{dt}\|u\|_Y^2 = 2\langle u', u \rangle \tag{2.2.28}$$

in the sense of scalar distributions on $(0, T)$.

An elementary proof may be found in [61], Lemma 1.2 of Chapter 3.
The spaces V and H from Section 2.1.2 are a typical example of the spaces X and Y.
Lemma 2.2.7 and Corollary 2.2.1 (with $E = \mathbb{R}$) imply

Corollary 2.2.3. *If some set F is bounded in $L_2(0, T; X)$, and the set $\{f' | f \in F\}$ is bounded in $L_2(0, T; X^*)$, then F is bounded in $C([0, T]; Y)$.*

2.2.9 Two lemmas on absolutely continuous vector functions

We finish this chapter with two technical lemmas. The first one is close to Lemma 3.1.1 from [61].

Lemma 2.2.8. *Let X be a Banach space, and let $u, g \in L_1(0, T; X)$. Let Z be an everywhere dense set in the $*$-weak topology of X^*. Then, the following four conditions are equivalent:*

i) *For each* $v \in Z$,

$$\frac{d}{dt}\langle v, u\rangle = \langle v, g\rangle \tag{2.2.29}$$

in the sense of scalar distributions on $(0, T)$*;*

ii) *for each* $v \in X^*$*, the distributional derivative* $\frac{d}{dt}\langle v, u(t)\rangle$ *is integrable on* $(0, T)$ *and almost everywhere equal to* $\langle v, g(t)\rangle$*;*

iii) $g = u'$ *in the sense of Definition 2.2.1;*

iv) $u \in C([0, T]; X)$*, and*

$$u(t) = u(0) + \int_0^t g(s)\, ds, \quad t \in [0, T]. \tag{2.2.30}$$

Proof. i) → ii). Let $v \in X^*$ and $v_m \in Z$, $v_m \underset{m\to\infty}{\rightharpoonup} v$ $*$-weakly. Then $\frac{d}{dt}\langle v_m, u\rangle = \langle v_m, g\rangle$. Passing to the limit in the sense of scalar distributions on $(0, T)$, we conclude that $\frac{d}{dt}\langle v, u\rangle = \langle v, g\rangle$. Since the right-hand side is integrable, the left-hand one is also integrable, so the equality holds almost everywhere on $(0, T)$.

ii) → iii). Take $\psi \in \mathcal{D}(0, T)$ and $v \in X^*$. Then

$$\begin{aligned}\langle v, u'(\psi)\rangle &= -\langle v, u(\psi')\rangle = -\Big\langle v, \int_0^T \psi'(t)u(t)\, dt\Big\rangle \\ &= -\int_0^T \langle v, u(t)\rangle\psi'(t)\, dt = \int_0^T \frac{d}{dt}\langle v, u(t)\rangle\psi(t)\, dt \\ &= \int_0^T \langle v, g(t)\rangle\psi(t)\, dt = \Big\langle v, \int_0^T g(t)\psi(t)\, dt\Big\rangle = \langle v, g(\psi)\rangle.\end{aligned}$$

Since $\psi \in \mathcal{D}(0, T)$ and $v \in X^*$ are arbitrary, $u' = g$.

iii) → iv) follows from Lemma 2.2.3.

iv) → ii). We have:

$$\frac{d}{dt}\langle v, u(t)\rangle = \frac{d}{dt}\Big\langle v, u(0) + \int_0^t g(s)\, ds\Big\rangle = \langle v, g(t)\rangle.$$

ii) → i) is clear. □

Lemma 2.2.9. *Let* E *be a reflexive Banach space,* $u, v \in C([0, T]; E)$*, and* $v' \in L_p(0, T; E)$*,* $1 < p < \infty$*. If there is a constant* C *such that for all* $t, s \in [0, T]$ *one has*

$$\|u(t) - u(s)\|_E \le C\|v(t) - v(s)\|_E, \tag{2.2.31}$$

then $u' \in L_p(0, T; E)$.

Proof. We shall use the notations τ_h and M_a from Section 2.2.3. Denote by u_h, $h > 0$, the function $\frac{1}{h}(\tau_h u - u)$. Fix a number $T_1 \in (\frac{T}{2}, T)$. Let $0 \le t \le T_1$. Then

$$\int_0^t u_h(s)\, ds = \frac{1}{h}\int_t^{t+h} u(s)\, ds - \frac{1}{h}\int_0^h u(s)\, ds \xrightarrow[h\to 0,\ T_1 \le T-h]{} u(t) - u(0) \tag{2.2.32}$$

in E.

The set $\{v'\}$ is compact in $L_p(0, T; E)$. As in the proof of Theorem 2.2.3 one shows that $M_h v' \xrightarrow[h\to 0]{} v'$ in $L_p(0, T_1; E)$. Let $h_k \xrightarrow[k\to\infty]{} 0$. Then the sequence $v_{h_k} = M_{h_k} v' = \frac{1}{h_k}(\tau_{h_k} v - v)$ is bounded in $L_p(0, T_1; E)$. But due to (2.2.31):

$$\|u_{h_k}\|_{L_p(0,T_1;E)} \le C\|v_{h_k}\|_{L_p(0,T_1;E)},$$

so $\{u_{h_k}\}$ is also bounded. Since E is reflexive, $L_p(0, T_1; E)$ is also reflexive. Therefore without loss of generality we may assume that there is $g \in L_p(0, T_1; E)$ such that $u_{h_k} \rightharpoonup g$ weakly in $L_p(0, T_1; E)$. The linear operator $l : w \mapsto \int_0^t w(s)\, ds$, $0 \le t \le T_1$, from $L_p(0, T_1; E)$ to E, is bounded (cf. the proof of Theorem 2.2.3). Hence,

$$\int_0^t u_{h_k}(s)\, ds \xrightarrow[k\to\infty]{} \int_0^t g(s)\, ds$$

weakly in E. But (2.2.32) gives

$$u(t) = u(0) + \int_0^t g(s)\, ds,$$

i.e.

$$u' = g \in L_p(0, T_1; E).$$

Repeating these arguments for the functions $\widetilde{u}(t) = u(T - t)$ and $\widetilde{v}(t) = v(T - t)$, we conclude that $\widetilde{u}' \in L_p(0, T_1; E)$, i.e. $u' \in L_p(T - T_1, T; E)$. Thus, $u' \in L_p(0, T; E)$. □

Chapter 3

Operator equations in Banach spaces

This chapter contains miscellaneous facts on linear and nonlinear non-evolutionary and evolutionary operator equations in Banach spaces, which are used later during the investigation of equations of hydrodynamics.

3.1 Linear equations

3.1.1 The Lax–Milgram theorem

The following projection theorem (see [61], Theorem 2.2 of Chapter 1) is a good tool to prove existence of solutions for linear equations (of *weak* solutions for elliptic equations, especially).

Theorem 3.1.1. *Let W be a separable real Hilbert space and let $b : W \times W \to \mathbb{R}$ be a continuous bilinear form. Let b be coercive, i.e. there is $\alpha > 0$ such that*

$$b(u,u) \geq \alpha \|u\|_W^2, \quad \forall u \in W. \tag{3.1.1}$$

Then, for each $g \in W^*$, there exists unique $u \in W$ such that

$$b(u,v) = \langle g, v \rangle, \quad \forall v \in W. \tag{3.1.2}$$

3.1.2 Characterization of the gradient of a distribution

The following profound result is due to de Rham and J.-L. Lions ([37], p. 67, see also [52], [61], [82], [56]).

Lemma 3.1.1. *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and let $g \in \mathcal{D}'(\Omega)^n$. Then,*

$$\langle g, v \rangle = 0 \textit{ for any } v \in C_0^\infty(\Omega)^n \textit{ with } \operatorname{div} v = 0 \tag{3.1.3}$$

if and only if there exists $q \in \mathcal{D}'(\Omega)$ such that

$$\operatorname{grad} q = g. \tag{3.1.4}$$

The basic regularity result for equation (3.1.4) is

Lemma 3.1.2 (see [41], p. 320). *Let Ω be a sufficiently regular bounded domain in $\mathbb{R}^n$. If $q \in \mathcal{D}'(\Omega)$ and* $\operatorname{grad} q \in H^{-1}(\Omega)^n$*, then $q \in L_2(\Omega)$.*

Corollary 3.1.1. *Assume that Ω is the union of a finite number of sufficiently regular bounded domains in $\mathbb{R}^n$. If $q \in \mathcal{D}'(\Omega)$ and* $\operatorname{grad} q \in H^{m-2}(\Omega)^n$*, $m \in \mathbb{N}$, then $q \in H^{m-1}(\Omega)$.*

Proof. Let $\Omega = \bigcup_{k=1}^{l} \Omega_k$ where Ω_k, $k = 1, \dots, l$, are sufficiently regular bounded domains. Let $q \in \mathcal{D}'(\Omega)$, and $g = \operatorname{grad} q \in H^{m-2}(\Omega)^n$, $m \in \mathbb{N}$. In particular, g belongs to $H^{-1}(\Omega)^n$, and its restriction $g|_{\Omega_k} \in H^{-1}(\Omega_k)^n$ can be defined as

$$\langle g|_{\Omega_k}, \varphi \rangle_{H^{-1}(\Omega_k)^n \times H_0^1(\Omega_k)^n} = \langle g, \widetilde{\varphi} \rangle_{H^{-1}(\Omega)^n \times H_0^1(\Omega)^n}$$

where $\varphi \in H_0^1(\Omega_k)^n$, and $\widetilde{\varphi} \in H_0^1(\Omega)^n$ coincides with φ almost everywhere on Ω_k and is identically zero outside Ω_k. Observe that $g|_{\Omega_k} = \operatorname{grad}(q|_{\Omega_k})$ where

$$\langle q|_{\Omega_k}, \phi \rangle_{\mathcal{D}'(\Omega_k) \times \mathcal{D}(\Omega_k)} = \langle q, \widetilde{\phi} \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)}.$$

Here $\phi \in \mathcal{D}(\Omega_k)$, and $\widetilde{\phi} \in \mathcal{D}(\Omega)$ coincides with ϕ on Ω_k and is identically zero outside Ω_k. By Lemma 3.1.2, $q|_{\Omega_k} \in L_2(\Omega_k)$ (for each k). Therefore $q \in L_2(\Omega)$. Since $\operatorname{grad} q \in H^{m-2}(\Omega)^n$, we conclude that $q \in H^{m-1}(\Omega)$. □

Corollary 3.1.2. *Let Ω be as in Corollary 3.1.1. Denote the connected components of Ω by Ω^k, $k = 1, 2, \dots$. Fix a function $\vartheta \in L_2(\Omega)$ with $\int_{\Omega^k} \vartheta(x)\, dx \neq 0$ for each k. Denote by $E_{\vartheta,\Omega}$ the subspace of $H^{m-1}(\Omega)$ which is the intersection of the kernels of the functionals $(\cdot, \vartheta)_{L_2(\Omega^k)}$. Denote by $H_G^{m-2}(\Omega)^n$ the subspace of $H^{m-2}(\Omega)^n$ which consists of the functions satisfying condition (3.1.3). Then the operator* grad *is an isomorphism of $E_{\vartheta,\Omega}$ onto $H_G^{m-2}(\Omega)^n$.*

Proof. The operator $\operatorname{grad} : E_{\vartheta,\Omega} \subset H^{m-1}(\Omega) \to H_G^{m-2}(\Omega)^n \subset H^{m-2}(\Omega)^n$ is continuous. Given $g \in H_G^{m-2}(\Omega)^n$, by Lemma 3.1.1 and Corollary 3.1.1, there exists $q \in H^{m-1}(\Omega)$ satisfying (3.1.4). Then the function q_* defined as

$$q_*(x) = q(x) - \frac{(q, \vartheta)_{L_2(\Omega^k)}}{\int_{\Omega^k} \vartheta(x)\, dx}, \quad x \in \Omega^k,$$

belongs to $E_{\vartheta,\Omega}$, and $\operatorname{grad} q_* = g$. Thus, the operator $\operatorname{grad} : E_{\vartheta,\Omega} \to H_G^{m-2}(\Omega)^n$ is surjective. It remains to show that its kernel is trivial. Let q belong to the kernel. For each k one has $\operatorname{grad} q(x) = 0$ at almost all $x \in \Omega^k$, so $q(x)$ does not depend on $x \in \Omega^k$ (due to connectedness of Ω^k). Hence, $0 = (q, \vartheta)_{L_2(\Omega^k)} = q\big|_{\Omega^k} \int_{\Omega^k} \vartheta(x)\, dx$ for each k, so $q \equiv 0$. □

Corollary 3.1.3. *Under the conditions of Corollary 3.1.2, the operator* grad *is an isomorphism of $Z(0, T; E_{\vartheta,\Omega})$ onto $Z(0, T; H_G^{m-2}(\Omega)^n)$ where the symbol Z stands for C, C^r $(r \in \mathbb{N})$ or W_p^r $(r \in \mathbb{Z}, 1 \le p \le \infty)$.*

The corollary follows straightforwardly from Corollary 3.1.2.

Now, let Z stand for C, C^r, L_p or W_p^r ($r \in \mathbb{N}, 1 \leq p \leq \infty$). For any domain $\Omega \subset \mathbb{R}^n$, let the symbol $Z(0, T; H^{m-1}_{\mathrm{loc}}(\Omega))$ ($m \in \mathbb{N}$) denote the set of functions u which

i) are defined almost everywhere on $(0, T)$,

ii) possess values in the set of Lebesgue measurable functions on Ω,

iii) for every open ball B such that the closure of B is contained in Ω, the restrictions $u|_B$ (defined as $u|_B(t) = u(t)|_B$ for almost all $t \in (0, T)$) belong to $Z(0, T; H^{m-1}(B))$.

Corollary 3.1.4. *Let Ω be a connected domain in $\mathbb{R}^n$. Let a function $\vartheta \in L_2(\Omega)$ have a compact support, and $\int_\Omega \vartheta(x)\ dx \neq 0$. Then for any $g \in Z(0, T; H_G^{m-2}(\Omega)^n)$ there exists a unique function $q \in Z(0, T; H^{m-1}_{\mathrm{loc}}(\Omega))$ such that $g(t) = \operatorname{grad} q(t)$ and $(q(t), \vartheta)_{L_2(\Omega)} = 0$ for almost all $t \in (0, T)$.*

Proof. Let $\mathcal{B}$ be the collection of all domains satisfying the properties

i) it may be represented as union of a finite number of sufficiently regular bounded domains;

ii) it is contained in Ω;

iii) it contains the support of ϑ;

iv) it is connected.

Let us show that for any open ball B such that its closure $\overline{B}$ is contained in Ω, there is a set from $\mathcal{B}$ which contains $\overline{B}$. Really, for any point $x \in \overline{B} \cup \operatorname{supp} \vartheta$ there is a ball $B(x) \subset \Omega$. Since $\overline{B} \cup \operatorname{supp} \vartheta$ is a compact set, its covering $\{B(x)\}$ has a finite subcovering $B(x_1), \ldots, B(x_k)$. Since Ω is an open connected set, any two balls from this subcovering can be connected with a broken line within Ω. The union of small regular neighbourhoods of these broken lines and of the balls $B(x_1), \ldots, B(x_k)$ belongs to $\mathcal{B}$ and contains $\overline{B}$.

Let $g \in Z(0, T; H_G^{m-2}(\Omega)^n)$. By Corollary 3.1.3, for any domain ω from $\mathcal{B}$, there is a unique function $q_\omega \in Z(0, T; H^{m-1}(\omega))$ such that $g(t) = \operatorname{grad} q_\omega(t)$ and $(q_\omega(t), \vartheta)_{L_2(\omega)} = 0$ for almost all $t \in (0, T)$. Then one can define the required function q by the formula $q(t)(x) = q_\omega(t)(x)$ where ω is a domain from $\mathcal{B}$ containing x. This definition is consistent, i.e. $q(t)(x)$ does not depend on the choice of ω. In fact, for any two domains $\omega_1, \omega_2 \in \mathcal{B}$ containing x, one has $\omega_1 \cup \omega_2 \in \mathcal{B}$, and $q_{\omega_1} = q_{\omega_1 \cup \omega_2}\big|_{\omega_1}$, $q_{\omega_2} = q_{\omega_1 \cup \omega_2}\big|_{\omega_2}$. Thus, $q_{\omega_1}(t)(x) = q_{\omega_1 \cup \omega_2}(t)(x) = q_{\omega_2}(t)(x)$. Note that $q|_B \in Z(0, T; H^{m-1}(B))$ for any open ball B such that its closure $\overline{B}$ is contained in Ω, so $q \in Z(0, T; H^{m-1}_{\mathrm{loc}}(\Omega))$. Uniqueness of each q_ω implies the uniqueness of q. □

Corollary 3.1.5. *Let Ω be an arbitrary domain in $\mathbb{R}^n$. Denote the connected components of Ω by Ω^k, $k = 1, 2, \ldots$. Let a function $\vartheta \in L_2(\Omega)$ have a compact support, and $\int_{\Omega^k} \vartheta(x)\, dx \neq 0$ for each k. Then for any $g \in Z(0,T; H_G^{m-2}(\Omega)^n)$ there exists a unique function $q \in Z(0,T; H_{\mathrm{loc}}^{m-1}(\Omega))$ such that $g(t) = \operatorname{grad} q(t)$ and $(q(t), \vartheta)_{L_2(\Omega^k)} = 0$ for each k and almost all $t \in (0,T)$.*

It results from application of Corollary 3.1.4 to each connected component of Ω.

3.1.3 An existence lemma

Assume that there are two Hilbert spaces, $X \subset Y$ with continuous embedding operator $i : X \to Y$, and $i(X)$ is dense in Y. Then, as in Subsection 2.2.8, we have (2.2.26) and (2.2.27).

The following lemma is a particular case of Theorem 1.1 from [20], Chapter VI.

Lemma 3.1.3. *Let X be separable and let $A : X \to X^*$ be a continuous linear operator. Assume that there is $\alpha > 0$ such that*

$$\langle Au, u\rangle \geq \alpha \|u\|_X^2, \quad \forall u \in X. \tag{3.1.5}$$

Then, given $a \in Y$ and $f \in L_2(0,T; X^*)$, the Cauchy problem

$$u'(t) + Au(t) = f(t), \quad t \in (0,T), \tag{3.1.6}$$

$$u(0) = a \tag{3.1.7}$$

possesses a unique solution in the class

$$u \in L_2(0,T; X), \quad u' \in L_2(0,T; X^*). \tag{3.1.8}$$

Remark 3.1.1. Due to Lemma 2.2.7, the solution u belongs to $C([0,T]; Y)$, so the initial condition (3.1.7) holds in the usual sense.

3.1.4 Strongly positive operators and parabolic equations

In this subsection we discuss some properties of abstract parabolic equations.

Let E be a Banach space. In the remainder of the chapter, we write $\|\cdot\|$ for the norm in E and for the norm of a bounded linear operator in E.

We consider the autonomous equation of the form:

$$\frac{dv}{dt} + \mathcal{A}(t)v = 0 \tag{3.1.9}$$

Here $v : [0,T] \to E$ is an unknown function; $\mathcal{A}(t) : D(\mathcal{A}) \subset E \to E$ is a linear unbounded operator for each $t \in [0,T]$.

We also consider the non-autonomous equation

$$\frac{dv}{dt} + \mathcal{A}(t)v = f(t) \tag{3.1.10}$$

where $f : [0,T] \to E$ is a given function.

Remark 3.1.2. The space E may be both complex and real. In the latter case, however, for issues concerning eigenvalues, complex numbers etc. one has to consider the complexification of the space E and the corresponding extension of the operator $\mathcal{A}$.

Definition 3.1.1 ([33]). A linear unbounded operator $\mathcal{B}$ with a dense domain in E is called *strongly positive* if for any complex number λ with Re $\lambda \geq 0$ the operator $\mathcal{B} + \lambda I$ has a bounded inverse operator and

$$\|(\mathcal{B} + \lambda I)^{-1}\| \leq \frac{K_1}{1 + |\lambda|}, \tag{3.1.11}$$

where the constant K_1 does not depend on λ.

A strongly positive operator $\mathcal{B}$ generates an analytical semigroup $e^{-\mathcal{B}t}$, $t \geq 0$, and it is possible to define its fractional powers (see [33, 58]):

$$\mathcal{B}^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \tau^{-\alpha-1} e^{-\tau\mathcal{B}} d\tau \quad (\alpha < 0), \tag{3.1.12}$$

where Γ is Euler's gamma-function;

$$\mathcal{B}^\alpha = (\mathcal{B}^{-\alpha})^{-1} \quad (\alpha > 0); \tag{3.1.13}$$

it is also assumed that $\mathcal{B}^0 = I$ (the identity operator). At $\alpha > 0$ the operators $\mathcal{B}^\alpha$ are not bounded. Their domains $D(\mathcal{B}^\alpha)$ are dense in E, and $D(\mathcal{B}^\alpha) \subset D(\mathcal{B}^\beta)$ for $\alpha > \beta$.

Theorem 3.1.2 (see [33], Theorem 14.8). *Let $\mathcal{B}_1$, $\mathcal{B}_2$ be two strongly positive operators in E. Assume that*

$$D(\mathcal{B}_1) \supset D(\mathcal{B}_2), \tag{3.1.14}$$

$$\|\mathcal{B}_1 u\| \leq K_2 \|\mathcal{B}_2 u\|, \; u \in D(\mathcal{B}_2), \tag{3.1.15}$$

where K_2 is independent of u. Then for any $0 \leq \varepsilon_1 < \varepsilon_2 \leq 1$,

$$D(\mathcal{B}_1^{\varepsilon_1}) \supset D(\mathcal{B}_2^{\varepsilon_2}), \tag{3.1.16}$$

$$\|\mathcal{B}_1^{\varepsilon_1} u\| \leq K_3(\varepsilon_1, \varepsilon_2) \|\mathcal{B}_2^{\varepsilon_2} u\|, \; u \in D(\mathcal{B}_2^{\varepsilon_2}), \tag{3.1.17}$$

where $K_3(\varepsilon_1, \varepsilon_2)$ is independent of u, $\mathcal{B}_1$, $\mathcal{B}_2$ but depends on K_2 (the notation $K_3(\varepsilon_1, \varepsilon_2)$ certainly means that K_3 depends on ε_1 and ε_2).

Definition 3.1.2. If an operator $\mathcal{A}(t)$ has a dense in E domain D which is independent of t, if it is strongly positive for every t, and if K_1 in (3.1.11) does not depend on t, we call $\mathcal{A}(t)$ *uniformly strongly positive* (with respect to t).

The following four results are due to Sobolevskii (see [58]).

Theorem 3.1.3. *Let an operator $\mathcal{A}(t)$ be uniformly strongly positive. Assume that for some $\varepsilon > 0$ and every $t, \tau, s \in [0, T]$ one has:*

$$\|(\mathcal{A}(t) - \mathcal{A}(\tau))\mathcal{A}^{-1}(s)\| \le C|t - \tau|^{\varepsilon} \tag{3.1.18}$$

(hereafter in this subsection C stands for various constants which do not depend on t, s, τ).

Then there exists a function $U(t, \tau)$ (the "resolving operator" for $\mathcal{A}$) with values in the space of linear bounded operators in E, which is defined and is strongly continuous on the triangle $0 \le \tau \le t \le T$. This operator function is uniformly differentiable with respect to t for $t > \tau$, and

$$\frac{\partial U(t, \tau)}{\partial t} + \mathcal{A}(t)U(t, \tau) = 0. \tag{3.1.19}$$

The following identities hold for $0 \le \tau \le s \le t \le T$:

$$U(t, \tau) = U(t, s)U(s, \tau), \tag{3.1.20}$$

$$U(t, t) = I. \tag{3.1.21}$$

For any $v_0 \in E$, the formula

$$v(t) = U(t, 0)v_0 \tag{3.1.22}$$

determines a solution of equation (3.1.9) which is continuous on $[0, T]$ and continuously differentiable for $t > 0$. If $v_0 \in D$, then $v \in C^1([0, T]; E)$.

Theorem 3.1.4. *Under the conditions of Theorem 3.1.3, the following estimates are valid:*

$$\|\mathcal{A}^{\alpha}(t)U(t, \tau)\mathcal{A}^{-\beta}(\tau)\| \le C|t - \tau|^{\beta-\alpha}, \tag{3.1.23}$$

where $0 \le \tau \le t \le T,\ 0 \le \beta \le \alpha < 1 + \varepsilon$;

$$\|\mathcal{A}^{\alpha}(0)(U(t + \Delta t, 0) - U(t, 0))\mathcal{A}^{-\beta}(0)\| \le C\Delta t^{\beta-\alpha}, \tag{3.1.24}$$

where $0 \le t \le t + \Delta t \le T,\ 0 \le \alpha \le \beta \le 1$.

Theorem 3.1.5. *Let $\mathcal{A}(t)$ satisfy the conditions of Theorem 3.1.3. Let $f \in C([0, T]; E)$ and let $\delta > 0$ be such that*

$$\|f(t) - f(\tau)\| \le C|t - \tau|^{\delta} \tag{3.1.25}$$

for all $t, \tau \in [0, T]$.

Given $v_0 \in E$, *the function*

$$v(t) = U(t,0)v_0 + \int_0^t U(t,s)f(s)ds \tag{3.1.26}$$

is continuous on $[0, T]$ *and continuously differentiable for* $t > 0$ *solution of equation (3.1.10). The function* $\mathcal{A}(t)v(t)$ *is continuous at* $t > 0$.

If $v_0 \in D$, *then*

$$v \in C^1([0,T]; E), \tag{3.1.27}$$

$$\mathcal{A}(\cdot)v \in C([0,T]; E). \tag{3.1.28}$$

If $v_0 \in D(\mathcal{A}^{1+\alpha}(0))$ *where* $\alpha < \min(\delta, \varepsilon)$, *and* $\mathcal{A}^{\alpha}(0)f \in C([0,T]; E)$, *then*

$$\mathcal{A}^{\alpha}(0)\mathcal{A}(t)v(t) \in C([0,T]; E). \tag{3.1.29}$$

Theorem 3.1.6. *Under the conditions of the Theorem 3.1.5, one has the following inequalities*

$$\begin{aligned}
&\left\| \mathcal{A}^{\alpha}(0) \left[\int_{\tau}^{t+\Delta t} U(t+\Delta t, s) f(s)\, ds - \int_{\tau}^{t} U(t,s) f(s)\, ds \right] \right\| \\
&\quad \le K_4(\alpha) \Delta t^{1-\alpha} (|\ln \Delta t| + 1) \|f\|_{C([\tau, t+\Delta t]; E)},
\end{aligned} \tag{3.1.30}$$

where $0 \le \tau \le t \le t + \Delta t \le T,\ 0 \le \alpha < 1$,

$$\left\| \frac{\partial}{\partial t} \int_{\tau}^{t} U(t,s) f(s) ds \right\| \le K_4' \cdot (\|f(t)\| + C), \tag{3.1.31}$$

where $0 \le \tau \le t \le T$ *and* C *is the constant from (3.1.25).*

Remark 3.1.3. The conditions of Theorem 3.1.3 may be replaced by the following one: the operator $\mathcal{A}(0)$ is strongly positive and for all $t, \tau \in [0, T]$

$$\|(\mathcal{A}(t) - \mathcal{A}(\tau))\mathcal{A}^{-1}(0)\| \le C|t-\tau|^{\varepsilon} \tag{3.1.32}$$

provided

$$(1 + K_1)CT^{\varepsilon} < 1, \tag{3.1.33}$$

where C and K_1 are the constants from (3.1.32) and estimate (3.1.11) for $\mathcal{A}(0)$. Let us show this. Take any complex λ with $\operatorname{Re}\lambda \ge 0$ and $s \in [0, T]$. We have:

$$\begin{aligned}
&\|([\mathcal{A}(s) + \lambda I] - [\mathcal{A}(0) + \lambda I])(\mathcal{A}(0) + \lambda I)^{-1}\| \\
&\quad \le \|(\mathcal{A}(s) - \mathcal{A}(0))\mathcal{A}^{-1}(0)\| \|\mathcal{A}(0)(\mathcal{A}(0) + \lambda I)^{-1}\| \\
&\quad \le \|(\mathcal{A}(s) - \mathcal{A}(0))\mathcal{A}^{-1}(0)\| \|I - \lambda(\mathcal{A}(0) + \lambda I)^{-1}\| \\
&\quad \le Cs^{\varepsilon}\big(1 + \frac{|\lambda| K_1}{1 + |\lambda|}\big) \le (1 + K_1)CT^{\varepsilon},
\end{aligned}$$

whence

$$\|[\mathcal{A}(s) + \lambda I](\mathcal{A}(0) + \lambda I)^{-1} - I\| \le (1 + K_1)CT^{\varepsilon} < 1.$$

As is well known, this yields that the operator $[\mathcal{A}(s)+\lambda I](\mathcal{A}(0)+\lambda I)^{-1}$ is invertible (and, hence, $\mathcal{A}(s) + \lambda I$ is also invertible), and we have the following bound for the norm of the inverse operator:

$$\|[\mathcal{A}(0) + \lambda I](\mathcal{A}(s) + \lambda I)^{-1}\| \le \frac{1}{1-(1+K_1)CT^{\varepsilon}}. \tag{3.1.34}$$

Thus, by estimate (3.1.11) for $\mathcal{A}(0)$,

$$\begin{aligned}
\|(\mathcal{A}(s) + \lambda I)^{-1}\| &\le \|[\mathcal{A}(0) + \lambda I]^{-1}\| \|[\mathcal{A}(0) + \lambda I](\mathcal{A}(s) + \lambda I)^{-1}\| \\
&\le \frac{K_1/(1+|\lambda|)}{1-(1+K_1)CT^{\varepsilon}},
\end{aligned}$$

so $\mathcal{A}(s)$ is uniformly strongly positive.

Using (3.1.32) and (3.1.34) for $\lambda = 0$, we obtain

$$\begin{aligned}
\|(\mathcal{A}(t) - \mathcal{A}(\tau))\mathcal{A}^{-1}(s)\| &\le \|(\mathcal{A}(t) - \mathcal{A}(\tau))\mathcal{A}^{-1}(0)\| \|\mathcal{A}(0)\mathcal{A}^{-1}(s)\| \\
&\le \frac{C}{1-(1+K_1)CT^{\varepsilon}}|t-\tau|^{\varepsilon}.
\end{aligned}$$

This estimate is similar to (3.1.18).

Remark 3.1.4. Let us show that (3.1.28) may be rewritten as

$$\mathcal{A}(0)v \in C([0,T]; E). \tag{3.1.35}$$

Really, for $0 \le t, t + \Delta t \le T$, one has

$$\begin{aligned}
\|\mathcal{A}(0)(v(t+\Delta t) - v(t))\| &\le \|\mathcal{A}(0)\mathcal{A}^{-1}(t)\| \, \|\mathcal{A}(t)(v(t+\Delta t) - v(t))\| \\
&\le \|\mathcal{A}(0)\mathcal{A}^{-1}(t)\| \, (\|(\mathcal{A}(t) - \mathcal{A}(t+\Delta t))v(t+\Delta t)\| \\
&\quad + \|\mathcal{A}(t+\Delta t)v(t+\Delta t) - \mathcal{A}(t)v(t)\|) \\
&\le C(\|(\mathcal{A}(t) - \mathcal{A}(t+\Delta t))\mathcal{A}^{-1}(t+\Delta t)\| \|\mathcal{A}(t+\Delta t)v(t+\Delta t)\| \\
&\quad + \|\mathcal{A}(t+\Delta t)v(t+\Delta t) - \mathcal{A}(t)v(t)\|) \\
&\le C|\Delta t|^{\varepsilon} + \|\mathcal{A}(t+\Delta t)v(t+\Delta t) - \mathcal{A}(t)v(t)\| \xrightarrow[\Delta t \to 0]{} 0.
\end{aligned}$$

Remark 3.1.5. Inequality (3.1.31) immediately implies

$$\begin{aligned}
&\Big\|\mathcal{A}(0)\int_0^t U(t,s)f(s)ds\Big\| \\
&\quad\le \|\mathcal{A}(0)\mathcal{A}^{-1}(t)\|\Big\|\mathcal{A}(t)\int_0^t U(t,s)f(s)ds\Big\| \\
&\quad= \|\mathcal{A}(0)\mathcal{A}^{-1}(t)\|\Big\|\frac{\partial}{\partial t}\int_0^t U(t,s)f(s)ds - f(t)\Big\| \\
&\quad\le C(K_4'\cdot(\|f(t)\|+C)+\|f(t)\|).
\end{aligned} \tag{3.1.36}$$

3.2 Nonlinear equations

3.2.1 An existence theorem

We will need the following existence theorem for nonlinear equations of parabolic type.

Theorem 3.2.1. a) *Let $\mathcal{B}: D(\mathcal{B}) \to E$ be a strongly positive operator in a Banach space E. Let α and R be some numbers, $0 \le \alpha < 1$, $R > 0$. Assume that for every $v \in D(\mathcal{B}^\alpha)$ such that $\|\mathcal{B}^\alpha v\| < R$ there is a linear operator $\mathcal{A}(v)(\cdot): D(\mathcal{B}) \to E$, and*

$$\|(\mathcal{A}(v)-\mathcal{A}(w))\mathcal{B}^{-1}\| \le K_5\|\mathcal{B}^\alpha(v-w)\|, \tag{3.2.1}$$

where K_5 does not depend on $v, w \in D(\mathcal{B}^\alpha)$ such that $\|\mathcal{B}^\alpha v\|,\ \|\mathcal{B}^\alpha w\| < R$.
Assume that for every v_0 such that

$$v_0 \in D(\mathcal{B}),\ \|\mathcal{B}^\alpha v_0\| < R, \tag{3.2.2}$$

the linear operator $\mathcal{A}_0 = \mathcal{A}(v_0)$ is strongly positive and the constant K_1 in (3.1.11) does not depend on v_0. Assume also that for all $v \in D(\mathcal{B})$:

$$K_6\|\mathcal{A}_0 v\| \le \|\mathcal{B}v\| \le K_7\|\mathcal{A}_0 v\|, \tag{3.2.3}$$

where $K_6,\ K_7$ do not depend on $v,\ v_0$.
Let $f_1: [0,T]\times D(\mathcal{B}^\alpha) \to E$ and

$$\|f_1(t,v)-f_1(s,w)\| \le K_8(\|\mathcal{B}^\alpha(v-w)\|+\|t-s\|), \tag{3.2.4}$$

where K_8 does not depend on t, s and $v, w \in D(\mathcal{B}^\alpha)$ such that $\|\mathcal{B}^\alpha v\|,\ \|\mathcal{B}^\alpha w\| < R$.

Then, for any v_0 satisfying (3.2.2), there exists a solution v of the problem

$$\frac{dv}{dt} + \mathcal{A}(v)v = f_1(t, v) \tag{3.2.5}$$

$$v(0) = v_0 \tag{3.2.6}$$

in the class

$$v \in C^1([0, t_0]; E), \ \mathcal{B}v \in C([0, t_0]; E), \tag{3.2.7}$$

where $t_0 = t_0(v_0)$ is some positive number.

b) *One can take $t_0 = T$ provided it is a priori known that the solution v is bounded in the following sense:*

$$\begin{aligned} \|\mathcal{B}^{\beta} v(t)\| &< K_9, \\ \|\mathcal{B}^{\alpha} v(t)\| &< K_9', \end{aligned} \tag{3.2.8}$$

where β is some fixed number, $\alpha < \beta \leq 1$, K_9 does not depend on t_0 and on $t \in [0, t_0]$, and $K_9' < R$.

Remark 3.2.1. As a matter of fact, this theorem is very close to a particular case of Theorem 7 from [58]. However, we give here the proof.

Proof. Take v_0 satisfying (3.2.2), and consider the set $Q = Q(t_0, \eta)$ which consists of the functions $v \in C([0, t_0]; E)$ with

$$v(0) = \mathcal{B}^{\alpha} v_0 \tag{3.2.9}$$

such that for any $t, \tau \in [0, t_0]$

$$\|v(t) - v(\tau)\| \leq |t - \tau|^{\eta} \tag{3.2.10}$$

(the parameters $t_0, \eta > 0$ will be defined later). The set Q is closed in $C([0, t_0]; E)$.

Since

$$\|v(0)\| = \|\mathcal{B}^{\alpha} v_0\| < R,$$

(3.2.10) implies

$$\|v(t)\| < R \tag{3.2.11}$$

for all $t \in [0, t_0]$, $v \in Q$, provided

$$t_0^{\eta} < R - \|\mathcal{B}^{\alpha} v_0\|.$$

Hence, for $v \in Q$ there is the operator function

$$\mathcal{A}_v(t) = \mathcal{A}(\mathcal{B}^{-\alpha} v(t)) \tag{3.2.12}$$

defined on $D(\mathcal{B})$ for all $t \in [0, t_0]$. Now, (3.2.1), (3.2.3) and (3.2.10) yield

$$\|(\mathcal{A}_v(t) - \mathcal{A}_v(\tau))\mathcal{A}_0^{-1}\| \leq \|(\mathcal{A}_v(t) - \mathcal{A}_v(\tau))\mathcal{B}^{-1}\| \, \|\mathcal{B}\mathcal{A}_0^{-1}\| \leq K_5 K_7 |t - \tau|^{\eta}. \tag{3.2.13}$$

By (3.2.9)

$$\mathcal{A}_v(0) = \mathcal{A}(v_0) = \mathcal{A}_0, \tag{3.2.14}$$

and $\mathcal{A}_0$ is strongly positive, so (3.2.13) implies (see Remark 3.1.3) that, for t_0 sufficiently small and $t \in [0, t_0]$, the operator $\mathcal{A}_v(t)$ is uniformly strongly positive. Furthermore,

$$\|(\mathcal{A}_v(t) - \mathcal{A}_v(\tau))\mathcal{A}_v^{-1}(s)\| \le K_{10}|t-\tau|^\eta \tag{3.2.15}$$

for all $t, \tau, s \in [0, t_0]$. By Theorem 3.1.3 there is a resolving operator $U_v(t,\tau)$ for $\mathcal{A}_v$. Then (3.1.24) gives

$$\|\mathcal{A}_0^{\alpha_1}(U_v(t+\Delta t, 0) - U_v(t,0))\mathcal{A}_0^{-\beta_1}\| \le K_{11}\Delta t^{\beta_1-\alpha_1} \tag{3.2.16}$$

for $0 \le t \le t + \Delta t \le t_0$, $0 \le \alpha_1 \le \beta_1 \le 1$.

Let $f_v(t) = f_1(t, \mathcal{B}^{-\alpha}v(t))$. Then (3.2.4) and (3.2.10) imply:

$$\|f_v(t) - f_v(\tau)\| \le K_{12}\|t-\tau\|^\eta. \tag{3.2.17}$$

Furthermore, by (3.2.4) and (3.2.11),

$$\begin{aligned} \|f_v(t)\| &\le \|f_1(t, \mathcal{B}^{-\alpha}v(t)) - f_1(0,0)\| + \|f_1(0,0)\| \\ &\le K_8(R+T) + \|f_1(0,0)\| = K_{13}. \end{aligned} \tag{3.2.18}$$

Since K_5, K_8, K_{10} do not depend on s, t, τ, v and v_0, the constants K_{11}, K_{12}, K_{13} are also independent of these variables.

By (3.1.30) we have the inequality:

$$\begin{aligned} &\left\|\mathcal{A}_0^{\alpha_1}\left[\int_0^{t+\Delta t} U_v(t+\Delta t, s) f_v(s)\,ds - \int_0^t U_v(t,s) f_v(s)\,ds\right]\right\| \\ &\quad \le K_{14}\Delta t^{1-\alpha_1}(|\ln \Delta t| + 1)\|f_v\|_{C([0,t_0];E)}, \quad 0 \le \alpha_1 < 1. \end{aligned} \tag{3.2.19}$$

Introduce the operator $w : Q \to C([0,t_0]; E)$ by the formula:

$$w(v)(t) = \mathcal{B}^\alpha U_v(t,0)v_0 + \mathcal{B}^\alpha \int_0^t U_v(t,s) f_v(s)\,ds. \tag{3.2.20}$$

Let $\alpha_1, \beta_1, \eta_1, \eta$ be numbers such that $\alpha < \alpha_1 < \alpha_1 + \eta < \alpha_1 + \eta_1 < \beta_1 < \beta$.

By (3.2.3) and Theorem 3.1.2 we have:

$$\|\mathcal{A}_0^{\gamma_1}u\| \le K_{15}(\gamma_1, \gamma)\|\mathcal{B}^\gamma u\|, \tag{3.2.21}$$

$$\|\mathcal{B}^{\gamma_1}u\| \le K_{16}(\gamma_1, \gamma)\|\mathcal{A}_0^\gamma u\| \tag{3.2.22}$$

for $0 < \gamma_1 < \gamma < 1$.

Using (3.2.16), (3.2.18) – (3.2.22) we get:

$$
\begin{aligned}
&\|w(v)(t+\Delta t)-w(v)(t)\| \\
&\quad \le \|\mathcal{B}^{\alpha}\mathcal{A}_0^{-\alpha_1}\|\Big(\|\mathcal{A}_0^{\alpha_1}(U_v(t+\Delta t,0)-U_v(t,0))\mathcal{A}_0^{-\beta_1}\|\|\mathcal{A}_0^{\beta_1}\mathcal{B}^{-\beta}\|\|\mathcal{B}^{\beta}v_0\| \\
&\qquad + \Big\|\mathcal{A}_0^{\alpha_1}\Big(\int_0^{t+\Delta t}U_v(t+\Delta t,s)f_v(s)\,ds-\int_0^{t}U_v(t,s)f_v(s)\,ds\Big)\Big\|\Big) \\
&\quad \le K_{17}(\Delta t^{\beta_1-\alpha_1}+\Delta t^{1-\alpha_1}(|\ln\Delta t|+1)) \le K_{18}\Delta t^{\eta}t_0^{\eta_1-\eta},
\end{aligned}
$$

where K_{18} depends only on $\|\mathcal{B}^{\beta}v_0\|$.

For t_0 small enough,

$$\|w(v)(t+\Delta t)-w(v)(t)\| \le \Delta t^{\eta}. \tag{3.2.23}$$

But

$$w(v)(0)=\mathcal{B}^{\alpha}v_0. \tag{3.2.24}$$

Therefore w transforms Q into itself.

Let us show that the map w is contracting. Let $v_1, v_2 \in Q$, and put $z_1 = \mathcal{B}^{-\alpha}w(v_1), z_2=\mathcal{B}^{-\alpha}w(v_2)$. It is easy to see that

$$z_1(0)=z_2(0)=v_0. \tag{3.2.25}$$

Theorem 3.1.5 gives that, for $t>0$, z_1, z_2 are continuously differentiable and

$$\frac{dz_i(t)}{dt}+\mathcal{A}_{v_i}(t)z_i(t)=f_{v_i}(t) \quad (i=1,2). \tag{3.2.26}$$

Hence,

$$\frac{d(z_1-z_2)}{dt}+\mathcal{A}_{v_1}(t)(z_1-z_2)=(\mathcal{A}_{v_2}(t)-\mathcal{A}_{v_1}(t))z_2+f_{v_1}(t)-f_{v_2}(t). \tag{3.2.27}$$

Let us show that

$$\|\mathcal{A}_0 z_2(t)\| \le K_{19}t^{\beta_1-1}. \tag{3.2.28}$$

We have:

$$\|\mathcal{A}_0 z_2(t)\| \le \|\mathcal{A}_0 U_{v_2}(t,0)v_0\| + \Big\|\mathcal{A}_0\int_0^t U_{v_2}(t,s)f_{v_2}(s)\,ds\Big\|. \tag{3.2.29}$$

Due to Remark 3.1.5 the second term in the right-hand side of (3.2.29) is simply bounded by a constant $K_{23} \le K_{24}t^{\beta_1-1}$. Putting in (3.2.15) $s=\tau$, $v=v_2$, we get

$$\|\mathcal{A}_{v_2}(t)\mathcal{A}_{v_2}^{-1}(\tau)\| \le \|\mathcal{A}_{v_2}(\tau)\mathcal{A}_{v_2}^{-1}(\tau)\| + K_{10}t_0^{\eta} \le K_{20}. \tag{3.2.30}$$

Hence, by Theorem 3.1.2,

$$\|\mathcal{A}_{v_2}^{\gamma_1}(t)\mathcal{A}_{v_2}^{-\gamma_2}(\tau)\| \le K_{21}, \quad \gamma_2 > \gamma_1. \tag{3.2.31}$$

With the help of (3.1.23), we can estimate now the first term in the right-hand side of (3.2.29):

$$\begin{aligned}\|\mathcal{A}_0 U_{v_2}(t,0)v_0\| &\le \|\mathcal{A}_0\mathcal{A}_{v_2}(t)^{-1}\|\|\mathcal{A}_{v_2}(t)U_{v_2}(t,0)\mathcal{A}_0^{-\beta_1}\|\|\mathcal{A}_0^{\beta_1}\mathcal{B}^{-\beta}\|\|\mathcal{B}^{\beta}v_0\|\\ &\le K_{22}t^{\beta_1-1}.\end{aligned}$$

Using (3.2.27) and (3.2.28) one can show (cf. [58], p. 341) that

$$z_1(t)-z_2(t) = \int_0^t U_{v_1}(t,s)[(\mathcal{A}_{v_2}(s)-\mathcal{A}_{v_1}(s))z_2(s)+(f_{v_1}(s)-f_{v_2}(s))]\,ds. \tag{3.2.32}$$

Applying to both sides the operator $\mathcal{B}^{\alpha}$ and taking into account (3.2.22), (3.1.23), (3.2.1), (3.2.28) and (3.2.4), we obtain:

$$\begin{aligned}&\|w(v_1)(t)-w(v_2)(t)\|\\ &\le \|\mathcal{B}^{\alpha}\mathcal{A}_0^{-\alpha_1}\|\int_0^t \|\mathcal{A}_0^{\alpha_1}U_{v_1}(t,s)\|[\|(\mathcal{A}_{v_2}(s)-\mathcal{A}_{v_1}(s))\mathcal{B}^{-1}\|\|\mathcal{B}\mathcal{A}_0^{-1}\|\|\mathcal{A}_0 z_2(s)\|\\ &\qquad + \|(f_{v_1}(s)-f_{v_2}(s))\|]\,ds\\ &\le K_{25}\int_0^t |t-s|^{-\alpha_1}\left(\|v_1(s)-v_2(s)\|s^{\beta_1-1}+\|v_1(s)-v_2(s)\|\right)ds\\ &\le K_{25}t_0^{\beta_1-\alpha_1}\|v_1-v_2\|_{C([0,t_0];E)}\int_0^t |t-s|^{-\beta_1}\left(s^{\beta_1-1}+1\right)ds,\end{aligned}$$

whence

$$\|w(v_1)-w(v_2)\|_{C([0,t_0];E)} \le K_{26}t_0^{\beta_1-\alpha_1}\|v_1-v_2\|_{C([0,t_0];E)}. \tag{3.2.33}$$

For t_0 small enough, $K_{26}t_0^{\beta_1-\alpha_1} < 1$, and w is a contraction. Thus, by the Banach principle, the map w has a fixed point $v_* \in Q$. Then the function

$$v(t) = \mathcal{B}^{-\alpha}v_*(t)$$

is a solution of the equation:

$$v(t) = U_{\mathcal{B}^{\alpha}v}(t,0)v_0 + \int_0^t U_{\mathcal{B}^{\alpha}v}(t,s)f_{\mathcal{B}^{\alpha}v}(s)\,ds. \tag{3.2.34}$$

By Theorem 3.1.5, $v(t)$ is a solution of (3.2.5) – (3.2.6). The linear operator $\mathcal{B}\mathcal{A}_0^{-1}$ is bounded, so (3.1.27) and (3.1.35) imply (3.2.7).

It remains to observe that the constants in the proof of the theorem, including t_0, are either independent of v_0 or depend on $\|\mathcal{B}^{\beta}v_0\|$ and on $R-\|\mathcal{B}^{\alpha}v_0\|$. Therefore, a priori bound (3.2.8) gives an opportunity to construct a solution, step by step, on the whole segment $[0,T]$. □

3.2.2 The Leray–Schauder degree

Here we recall some elementary facts from the Leray–Schauder degree theory [40, 32].

Let E be a normed space and B be the class of open bounded subsets of E. Denote by Ξ the set of triples $(I-k, D, p)$ where I is the identity operator in E, $p \in E$, $D \in B$, $k : \overline{D} \to E$ is a *compact operator* (i.e. k is continuous and its image is a relatively compact set), $p \notin (I-k)(\partial D)$. Here ∂D denotes the boundary of D.

Theorem 3.2.2. *There exists a unique map $d_{LS} : \Xi \to \mathbb{Z}$ satisfying the following four conditions (axioms).*

1) *(Normalization). For any $D \in B$ such that $0 \in D$, one has*

$$d_{LS}(I, D, 0) = 1.$$

2) *(Additivity). For any $(I-k, D, p) \in \Xi$ and all open sets $D_1, D_2 \subset D$ such that $p \notin f(\overline{D} \backslash (D_1 \bigcup D_2))$ one has*

$$d_{LS}((I-k)\big|_{\overline{D}_1}, D_1, p) + d_{LS}((I-k)\big|_{\overline{D}_2}, D_2, p) = d_{LS}(I-k, D, p).$$

3) *(Homotopic invariance). Let $D \in B$, $D \neq \varnothing$, and let $h : [0,1] \times \overline{D} \to E$ be a compact operator. Assume that $p \neq x - h(t,x)$ for $t \in [0,T], x \in \partial D$. Such h is called a homotopy. Then $d_{LS}(I-h_t, D, p)$ does not depend on $t \in [0,T]$ where $h_t : \overline{D} \to E$, $h_t(x) = h(t,x)$.*
4) *(Homogeneity). For any $(I-k, D, p) \in \Xi$ such that $D \neq \varnothing$, one has*

$$d_{LS}(f, D, p) = d_{LS}(f-p, D, 0).$$

Remark 3.2.2. Note that the fact that h is continuous and $h(t) : \overline{D} \to E$ is compact for all $t \in [0,T]$ does not imply that h is compact (see [32], p. 129). However, if $h_0, h_1 : \overline{D} \to E$ are compact, then the *linear homotopy* $h(t,x) = (1-t)h_0(x) + th_1(x) : [0,T] \times \overline{D} \to E$ is compact.

The map d_{LS} is called the *Leray–Schauder topological degree*. The most important property of the degree is

Theorem 3.2.3. *Let $(I-k, D, p) \in \Xi$ and $d_{LS}(I-k, D, p) \neq 0$. Then the equation*

$$x - k(x) = p \tag{3.2.35}$$

has a solution $x_0 \in D$.

A well-known consequence of this theorem is the Schauder fixed point principle. We shall use it in the following form:

Theorem 3.2.4. *Let A be a non-empty compact convex set in a normed space E. Then for any continuous map $\varphi : A \to A$ there is a point $x_* \in A$ such that $x_* = \varphi(x_*)$.*

Axioms 1) – 4) imply

Corollary 3.2.1. *For all $D \in B$ and $p \in D$,*

$$d_{LS}(I, D, p) = 1.$$

Chapter 4

Attractors of evolutionary equations in Banach spaces

4.1 Attractors of autonomous equations: classical approach

4.1.1 Attractor of a semigroup

Let E be an arbitrary set.

Definition 4.1.1. A family of mappings $\mathcal{S}_t : E \to E$, $t \geq 0$, is called a *semigroup* if $\mathcal{S}_0$ is the identity map I and

$$\mathcal{S}_t \circ \mathcal{S}_\tau = \mathcal{S}_{t+\tau} \tag{4.1.1}$$

for any $t, \tau \geq 0$.

Hereafter we assume that E is a Banach space.

Definition 4.1.2. A semigroup $\mathcal{S}_t$ is called *bounded* in E if, for any bounded set $B \subset E$, the set $\bigcup\limits_{t \geq 0} \mathcal{S}_t B$ is also bounded in E.

Let F be a topological space such that $E \cap F \neq \varnothing$.

Definition 4.1.3. A set $P \subset F$ is called (E, F)-*attracting* (for the semigroup $\mathcal{S}_t$) if for any bounded set $B \subset E$ and any open neighborhood W of P in F there exists $h \geq 0$ such that $\mathcal{S}_t B \subset W$ for all $t \geq h$.

Definition 4.1.4. A set $P \subset E$ is called *absorbing* (for the semigroup $\mathcal{S}_t$) if for any bounded set $B \subset E$ there is $h \geq 0$ such that for all $t \geq h$ one has $\mathcal{S}_t B \subset P$.

Definition 4.1.5. A set $A \subset E$ is called *invariant* (for the semigroup $\mathcal{S}_t$) if

$$\mathcal{S}_t A = A$$

for any $t \geq 0$.

Definition 4.1.6. A set $\mathcal{A} \subset E \cap F$ is called an (E, F)-*attractor* (of the semigroup $\mathcal{S}_t$) if

i) $\mathcal{A}$ is compact in F and bounded in E;

ii) $\mathcal{A}$ is invariant for the semigroup $\mathcal{S}_t$;

iii) $\mathcal{A}$ is (E, F)-attracting for the semigroup $\mathcal{S}_t$.

Various criteria for the existence of such an attractor may be found, for instance, in [27, 28, 9]. They are mainly based on the assumption that there is some (E, F)-attracting or absorbing set P, and the attractor (under some assumptions on the semigroup $\mathcal{S}_t$) can be found by the formula

$$\mathcal{A} = \bigcap_{s>s_0} \Big[\bigcup_{t\geq s} \mathcal{S}_t P\Big], \tag{4.1.2}$$

where s_0 is large enough and $[\cdot]$ stands for the closure in F.

Lemma 4.1.1. *Let $\mathcal{S}_t : E \to E$ be a semigroup. Let F be a T_1 space (i.e. the one-point sets in F are closed), and $E \cap F \neq \varnothing$. Let $A \subset E$ be bounded and an invariant set for the semigroup $\mathcal{S}_t$, and let $P \subset F$ be an (E, F)-attracting set for the semigroup $\mathcal{S}_t$. Then $A \subset P$.*

Proof. For any open neighborhood W of P in F, there is $h \geq 0$ such that $A = \mathcal{S}_t A \subset W$ for all $t \geq h$. Hence, $A \subset F$. If there is a point $x \in A$ such that $x \notin P$, then $W_x = F\backslash\{x\}$ is an open neighborhood of P in F. Therefore $A \subset W_x$, and we arrive at a contradiction. □

Corollary 4.1.1. *Let $\mathcal{S}_t : E \to E$ be a semigroup. Let F be a T_1 space , $E \cap F \neq \varnothing$. If there exists an (E, F)-attractor of the semigroup $\mathcal{S}_t$, it is unique.*

Proof. If there are two (E, F)-attractors $\mathcal{A}_1$ and $\mathcal{A}_2$, then, by Lemma 4.1.1, $\mathcal{A}_1 \subset \mathcal{A}_2$ and $\mathcal{A}_2 \subset \mathcal{A}_1$. □

4.1.2 Global (E, E_0)-attractors of evolutionary equations

Let E and E_0 be Banach spaces, $E \subset E_0$. Consider an abstract differential equation

$$\begin{gathered} u'(t) = A(u(t)), \\ u(t) \in E,\ A : D(A) \subset E \to R(A). \end{gathered} \tag{4.1.3}$$

The symbol "=" may be understood in any appropriate sense (e.g. in the sense of some topological space containing both E and $R(A)$). The derivative "′" may also be considered in any generalized sense. The nonlinear operator A is arbitrary (it may even be multi-valued, but in this case the symbol "=" must be replaced by "⊂"), but here we consider A to be independent of t (this means that the equation is "autonomous").

We shall investigate attractors of solutions of this equation which belong to the space $C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$.

Hereafter it is supposed that the space E is reflexive. Then, by Lemma 2.2.6, $C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)\subset C_w([0,+\infty);E)$. Hence, the values of functions from $C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$ belong to E at every $t\geq 0$.

Consider the translation (shift) operators $T(h)$,

$$T(h)(u)(t)=u(t+h),$$

$h\geq 0$ for $u\in C([0,+\infty);E_0)$, $L_\infty(0,+\infty;E)$, or $h\in\mathbb{R}$ for $C((-\infty,+\infty);E_0)$, $L_\infty(-\infty,+\infty;E)$.

For any fixed $h\geq 0$ the operators $T(h)$ are continuous bounded mappings of the spaces $C([0,+\infty);E_0)$ and $L_\infty(0,+\infty;E)$ into themselves.

Assume that for any $b\in E$ equation (4.1.3) possesses a unique solution u_b in a certain class $\mathcal{F}\subset C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$, satisfying the initial condition

$$u_b(0)=b. \tag{4.1.4}$$

Define the mapping $S_t:E\to E$, $t\geq 0$, by the formula

$$S_t(b)=u_b(t). \tag{4.1.5}$$

If

$$T(h)\mathcal{F}\subset\mathcal{F} \tag{4.1.6}$$

for all $h\geq 0$, then

$$T(h)u_b=u_{S_h(b)}. \tag{4.1.7}$$

This implies that S_t is a semigroup.

Definition 4.1.7. The (E,E_0)-attractor of the semigroup S_t is called the *global* (E,E_0)-*attractor* of the evolutionary equation (4.1.3).

4.2 Attractors of autonomous problems without uniqueness of the solution

In this section we describe a more general approach to attractors of equation (4.1.3), which does not require uniqueness of the solution to the Cauchy problem (4.1.3) – (4.1.4) and existence of the semigroup S_t.

4.2.1 Basic definitions

Let some set

$$\mathcal{H}^+ \subset C([0,+\infty);E_0) \cap L_\infty(0,+\infty;E)$$

of solutions (strong, weak, etc.) for equation (4.1.3) on the positive axis be fixed. The set $\mathcal{H}^+$ will be called the *trajectory space* and its elements will be called *trajectories*.

Remark 4.2.1. We do not assume that $T(h)\mathcal{H}^+ \subset \mathcal{H}^+$ for $h \geq 0$ (see also Remarks 4.2.5, 4.2.6).

Remark 4.2.2. Usually an appropriate trajectory space must be such that for every $a \in E$ there exists (but is not necessarily unique) a trajectory u satisfying the initial condition $u(0) = a$.

Remark 4.2.3. Below the concrete form of equation (4.1.3) is not significant but only presence of a trajectory space $\mathcal{H}^+$ is important and everything will depend only on the properties of this set. Generally speaking, the nature of $\mathcal{H}^+$ may be different from the one described above.

Definition 4.2.1. A set $P \subset C([0,+\infty);E_0) \cap L_\infty(0,+\infty;E)$ is called *attracting* (for the trajectory space $\mathcal{H}^+$) if for any set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(0,+\infty;E)$, one has

$$\sup_{u\in B} \inf_{v\in P} \|T(h)u - v\|_{C([0,+\infty);E_0)} \underset{h\to\infty}{\rightarrow} 0.$$

Definition 4.2.2. A set $P \subset C([0,+\infty);E_0) \cap L_\infty(0,+\infty;E)$ is called *absorbing* (for the trajectory space $\mathcal{H}^+$) if for any set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(0,+\infty;E)$, there is an $h \geq 0$ such that for all $t \geq h$:

$$T(t)B \subset P.$$

It is easy to see that any absorbing set is attracting.

Definition 4.2.3. A set $P \subset C([0,+\infty);E_0) \cap L_\infty(0,+\infty;E)$ is called a *trajectory semiattractor* (for the trajectory space $\mathcal{H}^+$) if

i) P is compact in $C([0,+\infty);E_0)$ and bounded in $L_\infty(0,+\infty;E)$;

ii) $T(t)P \subset P$ for any $t \geq 0$;

iii) P is attracting in the sense of Definition 4.2.1.

Definition 4.2.4. A set $P \subset C([0,+\infty);E_0) \cap L_\infty(0,+\infty;E)$ is called a *trajectory quasiattractor* (for the trajectory space $\mathcal{H}^+$) if it satisfies conditions i), iii) of Definition 4.2.3 and ii') $T(t)P \supset P$ for any $t \geq 0$.

Definition 4.2.5. A set $P \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ is called a *trajectory attractor* (for the trajectory space $\mathcal{H}^+$) if it is a trajectory semiattractor and a trajectory quasiattractor (for the trajectory space $\mathcal{H}^+$). A trajectory attractor is called *minimal* if it is contained in any other trajectory attractor.

Definition 4.2.6. A set $\mathcal{A} \subset E$ is called a *global attractor* (in E_0) for the trajectory space $\mathcal{H}^+$ of equation (4.1.3) if

i) $\mathcal{A}$ is compact in E_0 and bounded in E;

ii) for any set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(0,+\infty; E)$, the attraction property is fulfilled:

$$\sup_{u \in B} \inf_{v \in \mathcal{A}} \|u(t) - v\|_{E_0} \underset{t\to\infty}{\to} 0.$$

iii) $\mathcal{A}$ is the minimal set satisfying conditions i) and ii) (that is $\mathcal{A}$ is contained in every set satisfying conditions i) and ii)).

Remark 4.2.4. It is obvious that if there exists a minimal trajectory attractor or a global attractor, then it is unique.

Remark 4.2.5. If a trajectory attractor for the trajectory space $\mathcal{H}^+$ is contained in $\mathcal{H}^+$, then it is minimal. It follows from Lemma 4.2.10 (see below). In [66] (at the additional restriction that $T(h)\mathcal{H}^+ \subset \mathcal{H}^+$ for all $h \geq 0$) there were considered only trajectory attractors (in the sense of Definition 4.2.5) contained in $\mathcal{H}^+$. Nevertheless, the (more general) concept of minimal trajectory attractor used by us has many usual properties of trajectory attractors. In particular, a minimal trajectory attractor always generates a global attractor (see below, Theorem 4.2.2). Furthermore, under some conditions on the trajectory space $\mathcal{H}^+$ a minimal trajectory attractor (provided it exists) is always contained in $\mathcal{H}^+$ (see below, Remarks 4.2.11, 4.2.12).

Remark 4.2.6. A natural way to define a trajectory space is $\mathcal{H}^+$ is to take all the solutions (in a certain sense: strong, weak, etc.) for equation (4.1.3), which belong to some class $\mathcal{F} \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$. Usually, if u is a solution, then $T(h)u$, $h \geq 0$, is also a solution, so the condition $T(h)\mathcal{H}^+ \subset \mathcal{H}^+$ for all $h \geq 0$ may be violated only if (4.1.6) is violated. This may appear, for example, in the following situations:

a) $\mathcal{F}$ consists of the solutions satisfying some inequality which is not invariant with respect to the shifts $T(h)$. In particular, this may be an inequality containing integrals or other non-local functions, e.g. an inequality of the form

$$e^{Ct}\|u(t)\| \leq \max_{s\geq 0} \|u(s)\|, \quad t \geq 0.$$

Such issues appear at the study of the equations of motion for viscoelastic medium with Jeffreys' constitutive law (see Chapter 6).

b) $\mathcal{F}$ consists of the "surviving" solutions, i.e. the solutions u satisfying the property

$$u(t) \in M(u(0)), \quad t \in [0, T],$$

where $M : E \multimap E$ is a fixed multi-valued map and $T > 0$ is a fixed number.

c) $\mathcal{F}$ consists of the solutions which have some property at least at one point t or on some fixed subset of the positive axis, e.g. the solutions u for which there exists $t > 0$ such that $u(t) = u(0)$.

d) $\mathcal{F}$ consists of the solutions which do not have some property globally, e.g. all solutions except the ones satisfying the property

$$u(t) \in A, \quad t \geq 0,$$

where $A \subset E$ is a fixed set.

Let us introduce one more useful notion.

Definition 4.2.7. The *kernel* $\mathcal{K}(P)$ of a set $P \subset C([0, +\infty); E_0) \cap L_\infty(0, +\infty; E)$ is the set

$$\{u \in L_\infty(-\infty, +\infty; E) | \forall t \in \mathbb{R} : \Pi_+ T(t)u \in P\}.$$

Here Π_+ is the operator of restriction on the semi-axis $[0, +\infty)$.

Obviously, $\Pi_+ \mathcal{K}(P) \subset P$.

4.2.2 Simple properties of attracting sets and auxiliary statements

Lemma 4.2.1. *Let a set $P \subset C([0, +\infty); E_0) \cap L_\infty(0, +\infty; E)$ satisfy conditions i), iii) of Definition 4.2.3. Then $\Pi_+ \mathcal{K}(\mathcal{H}^+) \subset P$.*

Proof. Let $u \in \Pi_+ \mathcal{K}(\mathcal{H}^+)$, that is $u = \Pi_+ u_1$, where $u_1 \in \mathcal{K}(\mathcal{H}^+)$. Consider the set

$$B_u = \{\Pi_+ T(t) u_1 | t \in \mathbb{R}\}.$$

It suffices to show that $B_u \subset P$. The set B_u is contained in $\mathcal{H}^+$ and is bounded in $L_\infty(0, +\infty; E)$. Besides, $T(h) B_u \supset B_u$ for all $h \geq 0$. Really, if $v \in B_u$, then for some $t \in \mathbb{R}$ one has $v = \Pi_+ T(t) u_1 = T(h) \Pi_+ T(t - h) u_1 \in T(h) B_u$. Since P is an attracting set, for any $\varepsilon > 0$ there is an $h \geq 0$ such that

$$\sup_{w \in B_u} \inf_{v \in P} \|T(h)w - v\|_{C([0,+\infty); E_0)} < \varepsilon.$$

Since $T(h)B_u \supset B_u$,

$$\sup_{w \in B_u} \inf_{v \in P} \|w - v\|_{C([0,+\infty);E_0)} < \varepsilon.$$

The number $\varepsilon > 0$ was arbitrary, so

$$\inf_{v \in P} \|w - v\|_{C([0,+\infty);E_0)} = 0$$

for all $w \in B_u$. Since P is compact in $C([0,+\infty); E_0)$, $w \in P$. □

Roughly speaking, the kernel $\mathcal{K}(\mathcal{H}^+)$ is the set of solutions for equation (4.1.3) defined on the whole real axis, which are uniformly bounded in E and continuous with values in E_0. The following statement on properties of this set takes place.

Lemma 4.2.2. *Under the conditions of Lemma 4.2.1 the kernel $\mathcal{K}(\mathcal{H}^+)$ is relatively compact in $C((-\infty,+\infty); E_0)$ and bounded in $L_\infty(-\infty,+\infty; E)$.*

Proof. During the proof of the previous lemma it has been shown that the set

$$U = \{\Pi_+ T(t)u_1 | t \in \mathbb{R}, u_1 \in \mathcal{K}(\mathcal{H}^+)\}$$

is contained in P. Hence, U is contained in $C([0,+\infty); E_0)$ and is bounded in $L_\infty(0,+\infty; E)$. Therefore $\mathcal{K}(\mathcal{H}^+)$ is contained in $C((-\infty,+\infty); E_0)$ and is bounded in $L_\infty(-\infty,+\infty; E)$.

For $v \in \mathcal{K}(\mathcal{H}^+)$ and a natural number m we put

$$v_m(t) = v(t), \quad t \geq -m;$$
$$v_m(t) = v(-m), \quad t < -m.$$

Denote by $\mathcal{H}_m$ the set $\{v_m | v \in \mathcal{K}(\mathcal{H}^+)\}$. Since $\Pi_+\mathcal{K}(\mathcal{H}^+) \subset P$ is relatively compact in $C([0,+\infty); E_0)$, $\mathcal{H}_0$ is relatively compact in $C((-\infty,+\infty); E_0)$. It is easy to see that $T(m)\mathcal{K}(\mathcal{H}^+) = \mathcal{K}(\mathcal{H}^+)$. Therefore $\mathcal{H}_m = T(m)\mathcal{H}_0$ and it is relatively compact in $C((-\infty,+\infty); E_0)$. But for any $\varepsilon > 0$

$$\begin{aligned}\sup_{v \in \mathcal{K}(\mathcal{H}^+)} \inf_{w \in \mathcal{H}_m} \|w - v\|_{C((-\infty,+\infty);E_0)} &\leq \sup_{v \in \mathcal{K}(\mathcal{H}^+)} \|v_m - v\|_{C((-\infty,+\infty);E_0)} \\ &\leq \frac{1}{2^m} < \varepsilon\end{aligned}$$

for m large enough. Therefore $\mathcal{K}(\mathcal{H}^+)$ is relatively compact in $C((-\infty,+\infty); E_0)$ as a uniform limit of relatively compact sets $\mathcal{H}_m$ as $m \to \infty$ (Theorem 2.2.1 and Corollary 2.2.2 remain valid in Fréchet spaces: just replace norms with prenorms). □

Trajectory attractors possess the following interesting property.

Lemma 4.2.3. *If there exists a trajectory attractor P for the trajectory space $\mathcal{H}^+$, then $P = \Pi_+\mathcal{K}(P)$.*

Proof. As it was mentioned above, $P \supset \Pi_+\mathcal{K}(P)$. We shall show now the inverse inclusion. Let $u_0 \in P$. Since $T(1)P = P$, there exists $u_{-1} \in P$ such that $T(1)u_{-1} = u_0$. Similarly, there is $u_{-2} \in P$ such that $T(1)u_{-2} = u_{-1}$; $u_{-3} \in P$ such that $T(1)u_{-3} = u_{-2}$ and so on. Define $u \in L_\infty(-\infty, +\infty; E)$ as follows:

$$u(t) = u_0(t),\ t \geq 0; \quad u(t) = T(\{t\})u_{[t]}(0),\ t < 0.$$

Here the brackets $[\cdot]$ and $\{\cdot\}$ denote the integer part and the fractional part of a number, respectively. We shall show now that $u \in \mathcal{K}(P)$. Really, for $t \geq 0$, $\Pi_+T(t)u = T(t)u_0 \in T(t)P = P$, and, for $t < 0$, $\Pi_+T(t)u = T(\{t\})u_{[t]} \subset T(\{t\})P = P$. It remains to observe that $u_0 = \Pi_+u \in \Pi_+\mathcal{K}(P)$. □

Lemma 4.2.4. *a) Let sets $P_1, P_2 \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ satisfy conditions i) or ii) of Definition 4.2.3. Then $P_1 \cap P_2$ also satisfies a corresponding condition. b) If P_1, P_2 are compact in $C([0,+\infty); E_0)$ and satisfy condition iii) of Definition 4.2.3, then $P_1 \cap P_2$ also satisfies condition iii).*

Proof. Statement a) is clear. Let us show b). Let P_1, P_2 be compact in $C([0,+\infty); E_0)$ and satisfy condition iii). We have to show that $P_1 \cap P_2$ is an attracting set. If it is not so, then for some $\delta > 0$ and some set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(0,+\infty; E)$, there is a sequence $h_m \to \infty$ such that

$$\sup_{u \in B} \inf_{v \in P_1 \cap P_2} \|T(h_m)u - v\|_{C([0,+\infty);E_0)} > \delta.$$

Then there are elements $u_m \in B$ such that

$$\inf_{v \in P_1 \cap P_2} \|T(h_m)u_m - v\|_{C([0,+\infty);E_0)} > \delta. \tag{4.2.1}$$

On the other hand, since P_1 and P_2 are attracting sets, for any natural number k there exist a number m_k and elements $v_k^1 \in P_1$, $v_k^2 \in P_2$ such that

$$\|T(h_{m_k})u_{m_k} - v_k^1\|_{C([0,+\infty);E_0)} < \frac{1}{k},$$
$$\|T(h_{m_k})u_{m_k} - v_k^2\|_{C([0,+\infty);E_0)} < \frac{1}{k}.$$

Since P_1 is compact in $C([0,+\infty); E_0)$, without loss of generality we may assume that the sequence v_k^1 converges to an element v_0 as $k \to \infty$. Then sequences $T(h_{m_k})u_{m_k}$ and v_k^2 also converge to v_0. Thus, $v_0 \in P_1 \cap P_2$ and $\|T(h_{m_k})u_{m_k} - v_0\|_{C([0,+\infty);E_0)} \underset{k\to\infty}{\to} 0$, which contradicts (4.2.1). □

Lemma 4.2.5. *Let a set $P \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ satisfy one of conditions i), ii), ii') or iii) of Definitions 4.2.3, 4.2.4. Then $T(h)P$ also satisfies a corresponding condition for all $h \geq 0$.*

Proof. The statement of the lemma concerning conditions i), ii), ii') is obvious. Let P satisfy condition iii), that is it is attracting. Since the map $T(h)$ is bounded in $C([0,+\infty); E_0)$, one has

$$\|T(h)u\|_{C([0,+\infty);E_0)} \leq C\|u\|_{C([0,+\infty);E_0)}$$

for some constant C and all $u \in C([0,+\infty); E_0)$. Then for any set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(0,+\infty; E)$, one has

$$\begin{aligned} &\sup_{u\in B} \inf_{v\in T(h)P} \|T(t)u - v\|_{C([0,+\infty);E_0)} \\ &\quad = \sup_{u\in B} \inf_{v\in P} \|T(h)(T(t-h)u - v)\|_{C([0,+\infty);E_0)} \\ &\quad \leq C \sup_{u\in B} \inf_{v\in P} \|T(t-h)u - v\|_{C([0,+\infty);E_0)} \underset{t\to\infty}{\to} 0, \end{aligned}$$

that is $T(h)P$ is also attracting. □

We will need also the following statement.

Lemma 4.2.6. *Let (X,ρ) be a metric space and $\{K_\alpha\}_{\alpha\in\Xi}$ be a system of non-empty compact sets in X. Assume that for any $\alpha_1, \alpha_2 \in \Xi$ there is $\alpha_3 \in \Xi$ such that $K_{\alpha_1} \cap K_{\alpha_2} = K_{\alpha_3}$. Then $K_0 = \bigcap_{\alpha\in\Xi} K_\alpha \neq \varnothing$ and for any $\varepsilon > 0$ there is $\alpha_\varepsilon \in \Xi$ such that for any $y \in K_{\alpha_\varepsilon}$:*

$$\inf_{x\in K_0} \rho(x,y) < \varepsilon.$$

Proof. By induction one easily proves that an intersection of any finite number of sets from the system $\{K_\alpha\}$ belongs to this system.

Consider the set

$$K_\varepsilon = \{y \in X \mid \inf_{x\in K_0} \rho(x,y) < \varepsilon\} \quad \text{if} \quad K_0 \neq \varnothing;$$

$$K_\varepsilon = \varnothing \quad \text{if} \quad K_0 = \varnothing.$$

It is clear that K_ε is open and $K_0 \subset K_\varepsilon$. Note that if $K_\varepsilon \neq \varnothing$, then $K_0 \neq \varnothing$. Thus, to prove the lemma, it suffices to find a set $K_{\alpha_\varepsilon} \subset K_\varepsilon$ in the system $\{K_\alpha\}$.

Let $\alpha_0 \in \Xi$. Then $K_{\alpha_0}\backslash K_\varepsilon$ is compact and $\{X\backslash K_\alpha\}_{\alpha\in\Xi}$ is an open cover for $K_{\alpha_0}\backslash K_\varepsilon$. One can choose its finite subcover $X\backslash K_{\alpha_1}, X\backslash K_{\alpha_2}, \ldots, X\backslash K_{\alpha_m}$. Thus, $K_{\alpha_0}\backslash K_\varepsilon \subset X\backslash \bigcap_{i=1}^m K_{\alpha_i}$. Therefore $\bigcap_{i=1}^m K_{\alpha_i} \subset X\backslash(K_{\alpha_0}\backslash K_\varepsilon)$. It implies that the set $\bigcap_{i=0}^m K_{\alpha_i}$ is contained in K_ε, so it can be taken as the required set K_{α_ε}. □

By analogy to the concept of minimal trajectory attractor for the trajectory space $\mathcal{H}^+$ it is possible to introduce the concept of *minimal trajectory semiattractor* as a trajectory semiattractor contained in any other trajectory semiattractor.

Lemma 4.2.7. *A minimal trajectory semiattractor is always a minimal trajectory attractor.*

Remark 4.2.7. We will prove below the inverse statement (Lemma 4.2.8).

Proof. Let $\mathfrak{U}$ be a minimal trajectory semiattractor, that is it is a trajectory semiattractor and it is contained in any other trajectory semiattractor. By Lemma 4.2.5 $T(h)\mathfrak{U}$ is a trajectory semiattractor for all $h \geq 0$, therefore $\mathfrak{U} \subset T(h)\mathfrak{U}$. Thus, $\mathfrak{U}$ is a trajectory semiattractor and a trajectory quasiattractor, that is it is a (minimal) trajectory attractor. □

4.2.3 Existence of a minimal trajectory attractor

Theorem 4.2.1 (see [75, 76]). *Assume that there exists a trajectory semiattractor P for the trajectory space $\mathcal{H}^+$. Then there exists a minimal trajectory attractor $\mathfrak{U}$ for the trajectory space $\mathcal{H}^+$. Here one has*

$$\Pi_+\mathcal{K}(\mathcal{H}^+) \subset \mathfrak{U} = \Pi_+\mathcal{K}(\mathfrak{U}) \subset \Pi_+\mathcal{K}(P) \subset P$$

and the kernel $\mathcal{K}(\mathcal{H}^+)$ is relatively compact in $C((-\infty, +\infty); E_0)$ and bounded in $L_\infty(-\infty, +\infty; E)$.

Proof. Take in Lemma 4.2.6 $X = C([0, +\infty); E_0)$ and let $\{K_\alpha\}_{\alpha\in\Xi}$ be the system of all trajectory semiattractors for the trajectory space $\mathcal{H}^+$. Denote by $\mathfrak{U}$ the intersection of all trajectory semiattractors. By Lemma 4.2.4 the intersection of any two sets from the system $\{K_\alpha\}$ belongs to this system. Thus, for the system of semiattractors $\{K_\alpha\}$ the conditions of Lemma 4.2.6 hold.

Let us show that $\mathfrak{U}$ is a trajectory semiattractor. Clearly, $\mathfrak{U}$ satisfies conditions i) and ii) of Definition 4.2.3. We shall show now that $\mathfrak{U}$ satisfies condition iii), that is it is attracting. Fix $\varepsilon > 0$ and a set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(0, +\infty; E)$. By Lemma 4.2.6 there is a semiattractor P_ε such that for any $v \in P_\varepsilon$:

$$\inf_{w\in\mathfrak{U}} \|w - v\|_{C([0,+\infty);E_0)} < \frac{\varepsilon}{2}.$$

Since P_ε is an attracting set, there exists $h \geq 0$ such that for $t \geq h$:

$$\sup_{u\in B} \inf_{v\in P_\varepsilon} \|T(t)u - v\|_{C([0,+\infty);E_0)} < \frac{\varepsilon}{2}.$$

Therefore for every $u \in B$ there exists $v(u) \in P_\varepsilon$ such that

$$\|T(t)u - v(u)\|_{C([0,+\infty);E_0)} < \frac{\varepsilon}{2}.$$

We have:

$$
\begin{aligned}
&\sup_{u\in B}\inf_{w\in\mathfrak{U}}\|T(t)u-w\|_{C([0,+\infty);E_0)}\\
&\qquad\le \sup_{u\in B}(\|T(t)u-v(u)\|_{C([0,+\infty);E_0)}+\inf_{w\in\mathfrak{U}}\|v(u)-w\|_{C([0,+\infty);E_0)})\\
&\qquad\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}
$$

Thus, $\mathfrak{U}$ is a trajectory semiattractor. It is clear that it is minimal. By Lemma 4.2.7 $\mathfrak{U}$ is a minimal trajectory attractor. It remains to use Lemmas 4.2.1 – 4.2.3. □

Corollary 4.2.1. *Assume that there exists an absorbing set P for the trajectory space $\mathcal{H}^+$ which is compact in $C([0,+\infty);E_0)$ and bounded in $L_\infty(0,+\infty;E)$. Then there exists a minimal trajectory attractor $\mathfrak{U}$ for the trajectory space $\mathcal{H}^+$. Here one has $\Pi_+\mathcal{K}(\mathcal{H}^+)\subset\mathfrak{U}=\Pi_+\mathcal{K}(\mathfrak{U})\subset\Pi_+\mathcal{K}(P)$ and the kernel $\mathcal{K}(\mathcal{H}^+)$ is relatively compact in $C((-\infty,+\infty);E_0)$ and bounded in $L_\infty(-\infty,+\infty;E)$.*

Proof. Due to Theorem 4.2.1, it suffices to find a trajectory semiattractor $P_1\subset P$.

By Definition 4.2.2, for any set $B\subset\mathcal{H}^+$ which is bounded in $L_\infty(0,+\infty;E)$, there is a number $h(B)\ge 0$ such that for all $t\ge h(B)$ one has $T(t)B\subset P$. Denote by P_1 the closure in $C([0,+\infty);E_0)$ of the set

$$P_0=\bigcup_{B\in\mathcal{B}}\bigcup_{t\ge h(B)}T(t)B,$$

where $\mathcal{B}$ is the set of all sets contained in $\mathcal{H}^+$ which are bounded in $L_\infty(0,+\infty;E)$. The set P_1 is contained in P, so it is compact in $C([0,+\infty);E_0)$ and bounded in $L_\infty(0,+\infty;E)$. It is clear that it is absorbing. Moreover, $T(h)P_0\subset P_0$ for $h\ge 0$. This implies (cf. below, Lemma 4.2.9) that $T(h)P_1\subset P_1$. Thus, P_1 is a semiattractor. □

Lemma 4.2.8. *A minimal trajectory attractor is always a minimal trajectory semiattractor.*

Proof. Let $\mathfrak{U}$ be a minimal trajectory attractor and let P be a trajectory semiattractor. Since the minimal trajectory attractor is unique, by Theorem 4.2.1 $\mathfrak{U}\subset P$. Thus, $\mathfrak{U}$ is contained in any trajectory semiattractor, so it is a minimal trajectory semiattractor. □

4.2.4 Existence of a global attractor

Consider the sections of a trajectory attractor and the kernel at fixed $t\ge 0$:

$$\mathfrak{U}(t)=\{v(t)|v\in\mathfrak{U}\};$$

$$\mathcal{K}(P)(t) = \{v(t) | v \in \mathcal{K}(P)\}.$$

It is easy to see that these sets are contained in E (see Section 4.1.2).

Theorem 4.2.2 (see [75, 76]). *If there exists a minimal trajectory attractor $\mathcal{U}$ for the trajectory space $\mathcal{H}^+$, then there is a global attractor $\mathcal{A}$ for the trajectory space $\mathcal{H}^+$ of equation (4.1.3) and for all $t \geq 0$ one has*

$$\mathcal{K}(\mathcal{H}^+)(t) \subset \mathcal{A} = \mathcal{U}(t) = \mathcal{K}(\mathcal{U})(t).$$

Proof. Observe first that since $T(t)\mathcal{U} = \mathcal{U}$, $t \geq 0$, the set $\mathcal{A} = \mathcal{U}(t)$ does not depend on t. From Lemmas 4.2.1 and 4.2.3 it follows that $\mathcal{K}(\mathcal{H}^+)(t) \subset \mathcal{U}(t) = \mathcal{K}(\mathcal{U})(t)$. The set $\mathcal{U}$ is compact in $C([0, +\infty); E_0)$, so its section $\mathcal{A} = \mathcal{U}(0)$ is compact in E_0. Moreover, $\mathcal{U}$ is bounded in $L_\infty(0, +\infty; E)$, so for $u \in \mathcal{U}$ and almost all $t \geq 0$ (except those t which belong to some set $\mathcal{O}_u$ of zero measure)

$$\|u(t)\|_E \leq \sup_{u \in \mathcal{U}} \|u\|_{L_\infty(0,+\infty;E)}.$$

Since $u \in C_w([0, T]; E)$, one has $u(t) \to u(0)$ weakly in E as $t \to 0$. Thus,

$$\|u(0)\|_E \leq \liminf_{t \to 0, t \notin \mathcal{O}_u} \|u(t)\|_E \leq \sup_{u \in \mathcal{U}} \|u\|_{L_\infty(0,+\infty;E)},$$

so $\mathcal{A} = \mathcal{U}(0)$ is bounded in E.

Let a set $B \subset \mathcal{H}^+$ be bounded in $L_\infty(0, +\infty; E)$. Since $\mathcal{U}$ is an attracting set,

$$\sup_{u \in B} \inf_{v \in \mathcal{U}} \|T(h)u - v\|_{C([0,+\infty);E_0)} \underset{h \to \infty}{\to} 0.$$

It yields the pointwise convergence:

$$\sup_{u \in B} \inf_{v \in \mathcal{U}} \|(T(h)u - v)(t)\|_{E_0} \underset{h \to \infty}{\to} 0, \ t \geq 0.$$

At $t = 0$ we get

$$\sup_{u \in B} \inf_{v \in \mathcal{A} = \mathcal{U}(0)} \|u(h) - v\|_{E_0} \underset{h \to \infty}{\to} 0.$$

It remains to show that $\mathcal{A}$ is contained in every set $\mathcal{A}_0$ which is compact in E_0, bounded in E and possesses the attraction property

$$\sup_{u \in B} \inf_{v \in \mathcal{A}_0} \|u(t) - v\|_{E_0} \underset{t \to \infty}{\to} 0 \tag{4.2.2}$$

for every set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(0, +\infty; E)$. Let $\mathcal{U}_0 = \{u \in \mathcal{U} | u(t) \in \mathcal{A}_0 \ \forall t \geq 0\}$. It suffices to show that $\mathcal{U} \subset \mathcal{U}_0$. By Lemma 4.2.8 $\mathcal{U}$ is contained in every trajectory semiattractor. Hence, it is enough to show that $\mathcal{U}_0$ is a semiattractor. Since $\mathcal{U}_0 \subset \mathcal{U}$, $\mathcal{U}_0$ is relatively compact in $C([0, +\infty); E_0)$

and is bounded in $L_\infty(0,+\infty;E)$. For any sequence $\{u_m\} \subset \mathfrak{U}_0$ converging in $C([0,+\infty);E_0)$ the limit u_0 belongs to the (closed in $C([0,+\infty);E_0)$) set $\mathfrak{U}$. The convergence in $C([0,+\infty);E_0)$ yields the pointwise convergence: $u_m(t) \to u_0(t)$ in E_0 , $t \geq 0$. Since $\mathcal{A}_0$ is compact in E_0, $u_0(t) \in \mathcal{A}_0$, $t \geq 0$. Thus, $\mathfrak{U}_0$ is closed and compact in $C([0,+\infty);E_0)$. Since $T(t)\mathfrak{U} \subset \mathfrak{U}$, one has $T(t)\mathfrak{U}_0 \subset \mathfrak{U}_0$, $t \geq 0$. It remains to show that $\mathfrak{U}_0$ is an attracting set. If not, for some $\delta > 0$ and some set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(0,+\infty;E)$, there exists a sequence $h_m \to \infty$ such that

$$\sup_{u\in B} \inf_{v\in\mathfrak{U}_0} \|T(h_m)u - v\|_{C([0,+\infty);E_0)} > \delta.$$

Then there are elements $u_m \in B$ such that

$$\inf_{v\in\mathfrak{U}_0} \|T(h_m)u_m - v\|_{C([0,+\infty);E_0)} > \delta. \tag{4.2.3}$$

On the other hand, since $\mathfrak{U}$ is an attracting set, for every natural k there exist a number m_k and an element $v_k \in \mathfrak{U}$ such that

$$\|T(h_{m_k})u_{m_k} - v_k\|_{C([0,+\infty);E_0)} < \frac{1}{k}.$$

Since $\mathfrak{U}$ is compact in $C([0,+\infty);E_0)$, without loss of generality the sequence v_k converges to an element $v_0 \in \mathfrak{U}$ as $k \to \infty$. Then

$$\|T(h_{m_k})u_{m_k} - v_0\|_{C([0,+\infty);E_0)} \underset{k\to\infty}{\to} 0. \tag{4.2.4}$$

Now (4.2.4) together with (4.2.3) yield $v_0 \notin \mathfrak{U}_0$, that is $v_0(t_0) \notin \mathcal{A}_0$ for some $t_0 \geq 0$. But from (4.2.2) it follows that

$$\inf_{v\in\mathcal{A}_0} \|T(h_{m_k})u_{m_k}(t_0) - v\|_{E_0} \underset{k\to\infty}{\to} 0.$$

Then there is a sequence $\{v_k^*\} \subset \mathcal{A}_0$ such that

$$\|T(h_{m_k})u_{m_k}(t_0) - v_k^*\|_{E_0} \underset{k\to\infty}{\to} 0.$$

Since $\mathcal{A}_0$ is compact, without loss of generality v_k^* converges to some element v^*. But (4.2.4) gives:

$$\|T(h_{m_k})u_{m_k}(t_0) - v_0(t_0)\|_{E_0} \underset{k\to\infty}{\to} 0.$$

Therefore $v_0(t_0) = v^* \in \mathcal{A}_0$, and we have a contradiction. □

Theorems 4.2.1 and 4.2.2 give existence of a global attractor under the conditions of Theorem 4.2.1. It appears that existence of a global attractor may be proved also under weaker assumptions.

Definition 4.2.8. A trajectory quasiattractor $\mathcal{U}$ for the trajectory space $\mathcal{H}^+$ is called *homogeneous* if the set $\mathcal{U}(t)$ does not depend on $t \geq 0$.

Theorem 4.2.3. *Assume that there exists a attracting set P for the trajectory space $\mathcal{H}^+$ which is compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$. Then there exists a homogeneous trajectory quasiattractor $\mathcal{U} \subset P$ for the trajectory space $\mathcal{H}^+$ such that the set $\mathcal{A} = \mathcal{U}(t)$ is a global attractor for the trajectory space $\mathcal{H}^+$ of equation (4.1.3).*

Proof. Take in Lemma 4.2.6 $X = C([0,+\infty); E_0)$ and let $\{K_\alpha\}_{\alpha\in\Xi}$ be the system of all attracting sets for the trajectory space $\mathcal{H}^+$ which are compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$. Denote by $\mathcal{U}$ the intersection of all such sets. By Lemma 4.2.4 the intersection of any two sets from the system $\{K_\alpha\}$ belongs to this system. Thus, for system $\{K_\alpha\}$ the conditions of Lemma 4.2.6 hold.

Clearly, $\mathcal{U}$ satisfies condition i) of Definition 4.2.3. As in the proof of Theorem 4.2.1 one shows that $\mathcal{U}$ is an attracting set. By Lemma 4.2.5 for all $h \geq 0$, the set $T(h)\mathcal{U}$ is attracting, compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$, therefore $\mathcal{U} \subset T(h)\mathcal{U}$. Thus, $\mathcal{U}$ is a trajectory quasiattractor.

As in the proof of Theorem 4.2.2 one shows that $\mathcal{U}(0)$ satisfies conditions i), ii) of Definition 4.2.6. Since $\mathcal{U} \subset T(t)\mathcal{U}$, $\mathcal{U}(0) \subset \mathcal{U}(t)$ for all $t \geq 0$. It remains to show that $\mathcal{U}(t)$ is contained in any set $\mathcal{A}_0$ which satisfies conditions i), ii) of Definition 4.2.6 (in particular, in $\mathcal{U}(0)$). Denote $\mathcal{U}_0 = \{u \in \mathcal{U} | u(t) \in \mathcal{A}_0 \ \forall t \geq 0\}$. It suffices to show that $\mathcal{U} \subset \mathcal{U}_0$. Since $\mathcal{U}$ is contained in every set from the system $\{K_\alpha\}$, it is enough to show that $\mathcal{U}_0$ belongs to this system, that is it is an attracting set for the trajectory space $\mathcal{H}^+$ which is compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$. It may be realized in the same way as the check of the corresponding statements in the proof of Theorem 4.2.2. □

Remark 4.2.8. For facilitation of check of conditions of Theorems 4.2.1 and 4.2.3, and of Corollary 4.2.1, the following simple statement may be used.

Lemma 4.2.9. *Let P be an attracting (or absorbing) set for the trajectory space $\mathcal{H}^+$ which is relatively compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$. Then its closure $\overline{P}$ in $C([0,+\infty); E_0)$ is an attracting (resp. absorbing) set for the trajectory space $\mathcal{H}^+$ which is compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$. If in addition $T(t)P \subset P$ for any $t \geq 0$, then $\overline{P}$ is a semiattractor.*

Proof. Since P is bounded in $L_\infty(0,+\infty; E)$ and its elements are weakly continuous functions, there is a constant C such that for any function $u \in P$ one has $\|u(t)\|_E \leq C$ for $t \geq 0$. Let $u_0 \in \overline{P}$. Then in P there is a sequence $u_m \to u_0$ in $C([0,+\infty); E_0)$. Hence, $u_m(t) \to u_0(t)$ in E_0 for all $t \geq 0$. Since $\|u_m(t)\|_E \leq C$ and the space E is reflexive, $u_m(t) \rightharpoonup u_0(t)$ weakly in E and $\|u_0(t)\|_E \leq C$. Thus, $\overline{P}$ is bounded in $L_\infty(0,+\infty; E)$. It is easy to see that $\overline{P}$ is a compact attracting (absorbing) set in

$C([0,+\infty);E_0)$. Let $T(h)P \subset P$ for any $t \geq 0$. We shall show now that $T(h)\overline{P} \subset \overline{P}$. Let $u \in T(h)\overline{P}$. Then there are $u_0 \in \overline{P}$ and a sequence $u_m \in P$ such that $u_m \to u_0$, $u = T(h)u_0$. Then $T(h)u_m \to T(h)u_0 = u$, that is $u \in \overline{T(h)P} \subset \overline{P}$. Thus, $\overline{P}$ is a semiattractor. □

4.2.5 The case when a trajectory attractor is contained in the trajectory space

Trajectory attractors which are contained in $\mathcal{H}^+$ have additional properties.

Lemma 4.2.10. *Let $\mathcal{U} \subset \mathcal{H}^+$ be a trajectory quasiattractor for the trajectory space $\mathcal{H}^+$. Then*

a) *$\mathcal{U}$ is contained in any compact attracting set $P \subset C([0,+\infty);E_0)$;*

b) *in $\mathcal{H}^+$ there are no trajectory quasiattractors different from $\mathcal{U}$;*

c) *if in addition it is known that $T(h)\mathcal{U} \subset \mathcal{H}^+$ for all $h \geq 0$, then $\mathcal{U}$ is a minimal trajectory attractor.*

Proof. a) Since $\mathcal{U} \subset \mathcal{H}^+$ is bounded in $L_\infty(0,+\infty;E)$, for any neighborhood

$$P_\varepsilon = \{y \in C([0,+\infty);E_0) | \inf_{x\in P} \|x-y\|_{C([0,+\infty);E_0)} < \varepsilon\}$$

of the attracting set P in $C([0,+\infty);E_0)$ one has $\mathcal{U} \subset T(h)\mathcal{U} \subset P_\varepsilon$ at $h \geq 0$ large enough. Since P is compact in $C([0,+\infty);E_0)$, it yields $\mathcal{U} \subset P$.

b) If $\mathcal{U}_1 \subset \mathcal{H}^+$ is another trajectory quasiattractor, then by point a) one has $\mathcal{U}_1 \subset \mathcal{U}$, $\mathcal{U} \subset \mathcal{U}_1$.

c) By Lemma 4.2.5 $T(h)\mathcal{U}$ is a trajectory quasiattractor for all $h \geq 0$. If $T(h)\mathcal{U} \subset \mathcal{H}^+$, then by point b) $T(h)\mathcal{U} = \mathcal{U}$ and $\mathcal{U}$ is a trajectory attractor. By point a) it is minimal. □

Theorems 4.2.1 – 4.2.3 and Lemma 4.2.10 imply

Theorem 4.2.4. *Assume that there exists an attracting set P for the trajectory space $\mathcal{H}^+$ which is compact in $C([0,+\infty);E_0)$ and bounded in $L_\infty(0,+\infty;E)$. Let $T(h)P \subset \mathcal{H}^+$ for all $h \geq 0$. Then there exist a minimal trajectory attractor $\mathcal{U} = \Pi_+\mathcal{K}(\mathcal{H}^+)$ for the trajectory space $\mathcal{H}^+$ and a global attractor $\mathcal{A} = \mathcal{U}(t) = \mathcal{K}(\mathcal{H}^+)(t)$, $t \geq 0$ for the trajectory space $\mathcal{H}^+$ of equation (4.1.3).*

Remark 4.2.9. Under conditions of Theorem 4.2.4 $\mathcal{U} = \Pi_+\mathcal{K}(\mathcal{H}^+)$, so the kernel $\mathcal{K}(\mathcal{H}^+)$ is non-empty.

Remark 4.2.10. In [66] (Theorem 1.1) this theorem was proved under the additional assumption $T(h)\mathcal{H}^+ \subset \mathcal{H}^+$ for all $h \geq 0$. In this case the trajectory attractor can be constructed using an expression resembling formula (4.1.2):

$$\mathfrak{U} = \bigcap_{s>0} \Big[\bigcup_{t \geq s} T(t)P \Big], \tag{4.2.5}$$

where $[\cdot]$ stands for the closure in $C([0,+\infty); E_0)$.

4.2.6 Structure of the minimal trajectory attractor and of the homogeneous trajectory quasiattractor

The following theorem gives some characterization for the structure of a minimal trajectory attractor and helps to specify the connection of this concept with the trajectory attractor from [66].

Let us define a topology on the set $C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ as follows: a set $V \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ is closed if any sequence of elements from V which is bounded in $L_\infty(0,+\infty; E)$ and converges in $C([0,+\infty); E_0)$ has a limit that belongs to V. Hereafter the square brackets $[\cdot]$ denote the closure of a set in this topology.

Lemma 4.2.11. *For every fixed $h \geq 0$, the map*

$$T(h) : C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E) \to C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$$

is continuous. As a corollary, for any $V \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ one has

$$T(h)[V] \subset [T(h)V].$$

Proof. Let $W \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$, $[W] = W$. Take a sequence $\{u_m\} \subset T(h)^{-1}(W)$ which is bounded in $L_\infty(0,+\infty; E)$ and converges to a limit u_0 in $C([0,+\infty); E_0)$. Then the sequence $\{T(h)u_m\} \subset W$ is bounded in $L_\infty(0,+\infty; E)$ and converges to $T(h)u_0 \in W$ in $C([0,+\infty); E_0)$. Therefore $u_0 \in T(h)^{-1}(W)$. Thus, the set $T(h)^{-1}(W)$ is closed, so $T(h)$ is continuous.

Let $V \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$. Then $V \subset T(h)^{-1}([T(h)V])$. Since the set in the right-hand side is closed, $[V] \subset T(h)^{-1}([T(h)V])$. Hence, $T(h)[V] \subset [T(h)V]$. □

Theorem 4.2.5 (see [76]). *Assume that there exists a minimal trajectory attractor $\mathfrak{U}$ for the trajectory space $\mathcal{H}^+$. Then*

$$\Pi_+ \mathcal{K}(\mathcal{H}^+) \subset \mathfrak{U} \subset \Pi_+ \mathcal{K}\Big(\Big[\bigcup_{t \geq 0} T(t)\mathcal{H}^+ \Big]\Big). \tag{4.2.6}$$

Proof. The first inclusion is already proved (see Lemma 4.2.1). By Lemma 4.2.3 $\mathcal{U} = \Pi_+\mathcal{K}(\mathcal{U})$. Therefore it suffices to prove that $\mathcal{U} \subset [\bigcup_{t\geq 0} T(t)\mathcal{H}^+]$. By Lemma 4.2.8 $\mathcal{U}$ is contained in every trajectory semiattractor, so it suffices to show that $\mathcal{U} \cap [\bigcup_{t\geq 0} T(t)\mathcal{H}^+]$ is a semiattractor.

Observe first that $\mathcal{U} \cap [\bigcup_{t\geq 0} T(t)\mathcal{H}^+]$ is closed in $C([0,+\infty); E_0)$. But $\mathcal{U}$ is compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$, so $\mathcal{U} \cap [\bigcup_{t\geq 0} T(t)\mathcal{H}^+]$ satisfies condition i) of Definition 4.2.3.

We have for any $h \geq 0$:

$$T(h)\Big[\bigcup_{t\geq 0} T(t)\mathcal{H}^+\Big] \subset \Big[T(h)\bigcup_{t\geq 0} T(t)\mathcal{H}^+\Big] = \Big[\bigcup_{t\geq h} T(t)\mathcal{H}^+\Big] \subset \Big[\bigcup_{t\geq 0} T(t)\mathcal{H}^+\Big].$$

By Lemma 4.2.4 $\mathcal{U} \cap [\bigcup_{t\geq 0} T(t)\mathcal{H}^+]$ satisfies condition ii) of Definition 4.2.3.

It remains to show that $\mathcal{U} \cap [\bigcup_{t\geq 0} T(t)\mathcal{H}^+]$ satisfies condition iii) of Definition 4.2.3, that is it is attracting. If not, for some $\delta > 0$ and some set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(0,+\infty; E)$, there exists a sequence $h_m \to \infty$ such that

$$\sup_{u\in B} \inf_{v\in \mathcal{U}\cap[\bigcup_{t\geq 0} T(t)\mathcal{H}^+]} \|T(h_m)u - v\|_{C([0,+\infty);E_0)} > \delta.$$

Then there are elements $u_m \in B$ such that

$$\inf_{v\in \mathcal{U}\cap[\bigcup_{t\geq 0} T(t)\mathcal{H}^+]} \|T(h_m)u_m - v\|_{C([0,+\infty);E_0)} > \delta. \tag{4.2.7}$$

On the other hand, since $\mathcal{U}$ is an attracting set, for any natural k there are a number m_k and elements $v_k \in \mathcal{U}$ such that

$$\|T(h_{m_k})u_{m_k} - v_k\|_{C([0,+\infty);E_0)} < \frac{1}{k}.$$

Since $\mathcal{U}$ is compact in $C([0,+\infty); E_0)$, without loss of generality the sequence v_k converges to an element v_0 as $k \to \infty$. Then the sequence $T(h_{m_k})u_{m_k}$ also converges to v_0. This sequence is bounded in $L_\infty(0,+\infty; E)$ (since B is bounded in $L_\infty(0,+\infty; E)$) and it is contained in $\bigcup_{t\geq 0} T(t)\mathcal{H}^+$. Thus, $v_0 \in \mathcal{U} \cap [\bigcup_{t\geq 0} T(t)\mathcal{H}^+]$ and $\|T(h_{m_k})u_{m_k} - v_0\|_{C([0,+\infty);E_0)} \underset{k\to\infty}{\to} 0$, which contradicts (4.2.7). □

Remark 4.2.11. Let $T(h)\mathcal{H}^+ \subset \mathcal{H}^+$ at all $h \geq 0$ and $[\mathcal{H}^+] = \mathcal{H}^+$. If there exists a minimal trajectory attractor $\mathcal{U}$ for the trajectory space $\mathcal{H}^+$, then by Theorem 4.2.5 $\mathcal{U} = \Pi_+\mathcal{K}(\mathcal{H}^+) \subset \mathcal{H}^+$, that is $\mathcal{U}$ is a trajectory attractor in the sense of [66]. Using Theorem 4.2.2 or Theorem 4.2.4 we conclude that $\mathcal{A} = \mathcal{U}(t) = \mathcal{K}(\mathcal{H}^+)(t)$ $(t \geq 0)$ is a global attractor for the trajectory space $\mathcal{H}^+$ of equation (4.1.3).

Remark 4.2.12. In [66] there was investigated existence of attractors for the Navier–Stokes problem. The set of weak solutions of the Navier–Stokes problem satisfying an energy estimate of differential type was taken as a trajectory space $\mathcal{H}^+$. In this situation it appears that $T(h)\mathcal{H}^+ \subset \mathcal{H}^+$ for all $h \geq 0$ and $[\mathcal{H}^+] = \mathcal{H}^+$ ([66], Proposition 3.3).

Some similar information can be obtained under the conditions of Theorem 4.2.3 for the homogeneous trajectory quasiattractor.

Theorem 4.2.6. *Under the conditions of Theorem 4.2.3, the homogeneous trajectory quasiattractor $\mathcal{U}$ is contained in the set $[\bigcup_{t\geq 0} T(t)\mathcal{H}^+]$.*

Proof. Since $\mathcal{U}$ is contained in every attracting set for the trajectory space $\mathcal{H}^+$ which is compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$, it suffices to show that $\mathcal{U} \cap [\bigcup_{t\geq 0} T(t)\mathcal{H}^+]$ is an attracting set which is compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$. It may be realized in the same way as the check of the corresponding statements in the proof of Theorem 4.2.5. □

Corollary 4.2.2. *Under the conditions of Theorem 4.2.3, let $T(h)\mathcal{H}^+ \subset \mathcal{H}^+$ at all $h \geq 0$ and $[\mathcal{H}^+] = \mathcal{H}^+$. Then the homogeneous trajectory quasiattractor $\mathcal{U}$ is a minimal trajectory attractor for the trajectory space $\mathcal{H}^+$, and*

$$\mathcal{U} = \Pi_+\mathcal{K}(\mathcal{H}^+), \tag{4.2.8}$$

whereas for the global attractor one has the expression

$$\mathcal{A} = \mathcal{K}(\mathcal{H}^+)(t) \tag{4.2.9}$$

for any $t \geq 0$.

Proof. By Theorem 4.2.6, $\mathcal{U} \subset \mathcal{H}^+$. Moreover, $T(h)\mathcal{U} \subset T(h)\mathcal{H}^+ \subset \mathcal{H}^+$ for all $h \geq 0$. By Lemma 4.2.10, c), $\mathcal{U}$ is a minimal trajectory attractor, so (4.2.6) implies (4.2.8). Since $\mathcal{A} = \mathcal{U}(t)$, $t \geq 0$, one has (4.2.9). □

4.2.7 Correspondence between two concepts of global attractor

The results of the current subsection give some description of the connection between Definitions 4.1.7 and 4.2.6.

Let the assumptions of Section 4.1.2 hold true. Then we have the semigroup S_t. In this situation we can define the trajectory space as

$$\mathcal{H}_0^+ = \{u_b \in C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E) | u_b(t) = S_t b, b \in E\}. \quad (4.2.10)$$

Then (4.1.7) implies that

$$T(h)\mathcal{H}_0^+ \subset \mathcal{H}_0^+ \quad (4.2.11)$$

for all $h \geq 0$.

Lemma 4.2.12. *Any (E, E_0)-attracting set for the semigroup S_t has the attraction property from Definition 4.2.6, ii). If the semigroup S_t is bounded, then any set $P \subset E$ which is compact in E_0 and possesses the attraction property from Definition 4.2.6, ii), is (E, E_0)-attracting for the semigroup S_t.*

Proof. Let P be an (E, E_0)-attracting set for the semigroup S_t. Take a set $B \subset \mathcal{H}_0^+$ which is bounded in $L_\infty(0,+\infty; E)$. Due to (4.2.10), there is a bounded set $V \subset E$ such that

$$B = \{u_b \in C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E) | u_b(t) = S_t b, b \in V\}. \quad (4.2.12)$$

We have

$$\sup_{u \in B} \inf_{v \in P} \|u(t) - v\|_{E_0} = \sup_{b \in V} \inf_{v \in P} \|S_t b - v\|_{E_0}. \quad (4.2.13)$$

But P is (E, E_0)-attracting, so, for any $\varepsilon > 0$, $S_t V$ is contained in the open neighborhood of P in E_0 of the form

$$\{y \in E_0 | \inf_{v \in P} \|y - v\|_{E_0} < \varepsilon\} \quad (4.2.14)$$

for sufficiently large t. Therefore the right-hand side of (4.2.13) tends to zero as $t \to \infty$. Hence, P has the attraction property from Definition 4.2.6, ii).

Conversely, let $P \subset E$ possess the attraction property from Definition 4.2.6, ii). Take any bounded set $V \subset E$. If the semigroup S_t is bounded, the set B given by formula (4.2.12) is bounded in $L_\infty(0,+\infty; E)$, and (4.2.10) yields $B \subset \mathcal{H}_0^+$. Now the attraction property from Definition 4.2.6, ii), implies that the left-hand side of (4.2.13) tends to zero as $t \to \infty$. Take an open neighborhood W of P in E_0. Since P is compact in E_0, W contains a neighborhood of P of form (4.2.14) for some $\varepsilon > 0$. But the right-hand side of (4.2.13) tends to zero as $t \to \infty$, so the set $S_t V$ is contained in this neighborhood for sufficiently large t. Hence, P is (E, E_0)-attracting for the semigroup S_t. □

Lemmas 4.2.12 and 4.1.1 yield

Corollary 4.2.3. *If the semigroup S_t is bounded, then any set $P \subset E$ which is compact in E_0 and possesses the attraction property from Definition 4.2.6, ii), contains all invariant sets of the semigroup S_t that are bounded in E. In particular, a global attractor for the trajectory space $\mathcal{H}_0^+$ of equation (4.1.3) (in the sense of Definition 4.2.6) contains all invariant sets of the semigroup S_t that are bounded in E.*

Corollary 4.2.4. *Let the semigroup S_t be bounded. If there exists a global (E, E_0)-attractor $\mathcal{A}$ of equation (4.1.3), then it is a global attractor for the trajectory space $\mathcal{H}_0^+$ of equation (4.1.3).*

Proof. Property i) of Definition 4.2.6 is fulfilled for $\mathcal{A}$, and property ii) follows from Lemma 4.2.12. Observe that property iii) also holds. Really, for any set $P \subset E$ satisfying properties i) and ii) of Definition 4.2.6, one has $\mathcal{A} \subset P$ by Corollary 4.2.3. □

The reciprocal connection between Definitions 4.1.7 and 4.2.6 is more delicate.

Corollary 4.2.5 (cf. [66], Corollary 2.1). *Let the conditions of Theorem 4.2.2 hold for the trajectory space $\mathcal{H}_0^+$. Let the semigroup S_t be bounded. If the minimal trajectory attractor $\mathcal{U}$ is contained in $\mathcal{H}_0^+$, then the global attractor $\mathcal{A} = \mathcal{U}(t)$, $t \geq 0$, is a global (E, E_0)-attractor of equation (4.1.3).*

Proof. In view of Lemma 4.2.12 it suffices to prove that $\mathcal{A}$ is invariant for the semigroup S_t.

The inclusion $\mathcal{U} \subset \mathcal{H}_0^+$ and formula (4.2.10) involve the representation:

$$\mathcal{U} = \{u_b \in C([0, +\infty); E_0) \cap L_\infty(0, +\infty; E) | u_b(t) = S_t b, b \in \mathcal{U}(0)\}. \quad (4.2.15)$$

Hence, for all $h \geq 0$, $S_h \mathcal{U}(0) = \mathcal{U}(h)$, i.e. $S_h \mathcal{A} = \mathcal{A}$. □

Corollaries 4.2.5 and 4.2.2 imply

Corollary 4.2.6. *Let the conditions of Theorem 4.2.3 hold for the trajectory space $\mathcal{H}_0^+$. Let the semigroup S_t be bounded. If $\mathcal{H}_0^+ = [\mathcal{H}_0^+]$, then the global attractor $\mathcal{A}$ (existing by Theorem 4.2.3) is a global (E, E_0)-attractor of equation (4.1.3).*

Definition 4.2.9. We say that the semigroup S_t is *closed* if for any $t > 0$ and any sequence $\{b_m\}$ bounded in E such that

i) $b_m \to b_0 \in E$ in the topology of E_0,

ii) $S_t b_m$ converges in E_0, one has $\lim_{m\to\infty} S_t b_m = S_t b_0$.

Corollary 4.2.7. *Let the conditions of Theorem 4.2.3 hold for the trajectory space $\mathcal{H}_0^+$. Let the semigroup S_t be bounded and closed. Then the global attractor $\mathcal{A}$ is a global (E, E_0)-attractor of equation (4.1.3).*

Proof. In view of Corollary 4.2.6 it is enough to show that $[\mathcal{H}_0^+] = \mathcal{H}_0^+$. Take any sequence of trajectories $v_m \in \mathcal{H}_0^+$, $v_m \to v_0$ in $C([0,+\infty); E_0)$ which is bounded in $L_\infty(0,+\infty; E)$. Then $v_m(t) = S_t b_m$, $b_m \in E$, $m \in \mathbb{N}$, $t \geq 0$. But $v_m(t) \to v_0(t)$ in E_0. Let $b_0 = v_0(0)$. Then $b_m = v_m(0) \to b_0$, $S_t b_m \to v_0(t)$ in E_0, so $v_0(t) = S_t b_0$, i.e. $v_0 \in \mathcal{H}_0^+$. □

Remark 4.2.13. If the semigroup S_t is not closed (but still bounded), the global (E, E_0)-attractor may not exist even if there exist a minimal trajectory attractor and a global attractor for the trajectory space $\mathcal{H}_0^+$. Such a situation appears in the following example. In the space $E = E_0 = \mathbb{R}$, consider the following combined differential equation:

$$\begin{aligned} &u'(t) = -(u(t)-1)^2, \quad u(t) > 1, \\ &u'(t) = -u^2(t), \quad 0 \leq u(t) \leq 1, \\ &u'(t) = u^2(t), \quad u(t) < 0. \end{aligned} \tag{4.2.16}$$

The semigroup S_t here acts as follows:

$$\begin{aligned} &S_t(0) \equiv 0, \\ &S_t b = \Big(t + \frac{1}{b-1}\Big)^{-1} + 1, \quad b > 1, \\ &S_t b = \Big(t + \frac{1}{b}\Big)^{-1}, \quad 0 < b \leq 1, \\ &S_t b = -\Big(t - \frac{1}{b}\Big)^{-1}, \quad b < 0. \end{aligned} \tag{4.2.17}$$

The minimal trajectory attractor is the set $\{u_0(t) \equiv 0, u_1(t) \equiv 1\}$. Hence, the global attractor is the set $\{0, 1\}$. However, there is no global $(\mathbb{R}, \mathbb{R})$-attractor, since the only bounded invariant set is $\{0\}$, but it is not $(\mathbb{R}, \mathbb{R})$-attracting for the semigroup S_t.

4.3 Attractors of non-autonomous equations

Let E and E_0 be Banach spaces, $E \subset E_0$, E is reflexive. Following [14, 15], we write an abstract non-autonomous evolution differential equation in the form

$$u'(t) = A_{\sigma(t)}(u(t)), \quad u(t) \in E. \tag{4.3.1}$$

Here σ is a functional parameter, which is called the time symbol of equation (4.3.1). Assume that σ belongs to some fixed parameter set Σ, which is called the *symbol space* and is usually a subset of some space of time-dependent functions. In applications, a function $\sigma(t)$ consists of all time-dependent coefficients, terms and right-hand sides of a considered equation.

We investigate attractors of solutions of equation (4.3.1) which belong to the space $C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$.

Assume that for every $\sigma \in \Sigma$ we have a fixed set

$$\mathcal{H}_\sigma^+ \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$$

of solutions (strong, weak, etc.) for equation (4.3.1), defined on the positive axis $t \geq 0$. The sets $\mathcal{H}_\sigma^+$ are called *trajectory spaces* and their elements are called *trajectories*.

Consider the united trajectory space $\mathcal{H}_\Sigma^+ = \bigcup\limits_{\sigma\in\Sigma} \mathcal{H}_\sigma^+$.

Definition 4.3.1. A set $P \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ is called *uniformly* (with respect to $\sigma \in \Sigma$) *attracting* (for equation (4.3.1)) if for any set $B \subset \mathcal{H}_\Sigma^+$ which is bounded in $L_\infty(0,+\infty; E)$, one has

$$\sup_{u\in B} \inf_{v\in P} \|T(h)u - v\|_{C([0,+\infty);E_0)} \underset{h\to\infty}{\to} 0.$$

Definition 4.3.2. A set $P \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ is called *uniformly absorbing* if for any set $B \subset \mathcal{H}_\Sigma^+$, which is bounded in $L_\infty(0,+\infty; E)$, there is $h \geq 0$ such that for all $t \geq h$:

$$T(t)B \subset P.$$

It is easy to see that any uniformly absorbing set is uniformly attracting.

Definition 4.3.3. A set $P \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ is called a *uniform trajectory semiattractor* (for equation (4.3.1)) if

i) P is compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$;

ii) $T(t)P \subset P$ for any $t \geq 0$;

iii) P is uniformly attracting in the sense of Definition 4.3.1.

Definition 4.3.4. A set $P \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ is called a *uniform trajectory quasiattractor* (for equation (4.3.1)) if it satisfies conditions i), iii) of Definition 4.3.3 and

ii') $T(t)P \supset P$ for any $t \geq 0$.

Definition 4.3.5. A uniform trajectory quasiattractor $\mathcal{U}$ is called *homogeneous* if the set $\mathcal{U}(t)$ does not depend on $t \geq 0$.

Definition 4.3.6. A set $P \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ is called a *uniform* (with respect to $\sigma \in \Sigma$) *trajectory attractor* (for equation (4.3.1)) if it is a uniform trajectory semiattractor and a uniform trajectory quasiattractor (i.e. $T(t)P = P$).

Definition 4.3.7. A uniform trajectory attractor $\mathfrak{U}$ is called *minimal* if it is contained in any other uniform trajectory attractor.

Definition 4.3.8. A set $\mathcal{A} \subset E$ is called a *uniform* (with respect to $\sigma \in \Sigma$) *global attractor* (in E_0) for equation (4.3.1) if

i) $\mathcal{A}$ is compact in E_0 and bounded in E;

ii) for any set $B \subset \mathcal{H}^+_\Sigma$ which is bounded in $L_\infty(0,+\infty; E)$, the attraction property is fulfilled:
$$\sup_{u\in B}\inf_{v\in\mathcal{A}} \|u(t) - v\|_{E_0} \underset{t\to\infty}{\to} 0.$$

iii) $\mathcal{A}$ is the minimal set satisfying conditions i) and ii) (that is $\mathcal{A}$ is contained in every set satisfying conditions i) and ii)).

Remark 4.3.1. The minimal uniform trajectory attractor and the uniform global attractor depend on the symbol space Σ. It is easy to observe that if we have two symbol spaces $\Sigma_1 \subset \Sigma_2$, then for corresponding minimal uniform trajectory attractors and uniform global attractors one has $\mathfrak{U}_{\Sigma_1} \subset \mathfrak{U}_{\Sigma_2}$, $\mathcal{A}_{\Sigma_1} \subset \mathcal{A}_{\Sigma_2}$. Moreover, existence of uniform attractors may fail after an extension of the symbol space (see Remark 4.3.3).

Remark 4.3.2. We do not assume that the semigroup $T(t)$ *acts* on Σ, i.e. that for any $t \geq 0$ the map $T(t) : \sigma(s) \to \sigma(s+t)$ (remember that the time symbols σ are time-dependent functions) transforms Σ into itself, and the family $T(t)$ is a semigroup on Σ in the sense of Definition 4.1.1. We do not assume also that the family of trajectory spaces $\mathcal{H}^+_\sigma$, $\sigma \in \Sigma$, is *translation-coordinated*, i.e. $T(h)\mathcal{H}^+_\sigma \subset \mathcal{H}^+_{T(h)\sigma}$ for $h \geq 0$ (see [15]).

Remark 4.3.3. Since the action of the semigroup $T(t)$ is not required of the symbol space Σ, one may assume that the symbol space consists of only one element σ_* corresponding to the particular equation under consideration. However, in applications, in order to justify the word *uniform* in Definitions 4.3.1 – 4.3.8 it seems to be preferable to assume that $\Sigma \supset \Sigma_0$ where
$$\Sigma_0 = \{T(t)\sigma_* | t \geq 0\}.$$
But the uniform attractors in the case $\Sigma = \{\sigma_*\}$ (as well as in the case of any other possible Σ) are of independent interest. Let us illustrate this with the following example. In the space $E = E_0 = \mathbb{R}$ we consider the differential equation
$$u'(t) = \frac{u(t)}{t}\,\chi(u^2(t)t^2). \tag{4.3.2}$$

Here $\chi(s) = 1$ for $s > 2$, $\chi(s) = 0$ for $s < 1$ and $\chi(s) = -1$ elsewhere. The solutions of this equation are

$$u(t) = Ct, \quad u^2(t)t^2 > 2, \tag{4.3.3}$$

$$u(t) = \frac{C}{t}, \quad 1 \le u^2(t)t^2 \le 2, \tag{4.3.4}$$

$$u(t) = C, \quad u^2(t)t^2 < 1. \tag{4.3.5}$$

Here and below in this remark, C is a real number.

Let us rewrite (4.3.2) in the form (4.3.1):

$$u'(t) = u(t)\sigma(t) \tag{4.3.6}$$

where $\sigma(t)$ is a scalar function of $t > 0$. Equation (4.3.2) corresponds to the case

$$\sigma(t) = \sigma_*(t) = \frac{\chi(u^2(t)t^2)}{t}.$$

For each function σ, define the trajectory space $\mathcal{H}_\sigma^+$ as the subset of $C([0, +\infty); \mathbb{R})$ consisting of the functions which at almost all $t > 0$ are differentiable and satisfy (4.3.6). Note that the trajectories from $\mathcal{H}_{\sigma_*}^+$ are of the form

$$u(t) = C, \quad t|C| < 1, \tag{4.3.7}$$

$$u(t) = \frac{\operatorname{sign} C}{t}, \quad t|C| \ge 1. \tag{4.3.8}$$

Therefore, if $\Sigma = \{\sigma_*\}$, then the minimal uniform trajectory attractor is the set $\{u_0(t) \equiv 0\}$, and the uniform global attractor is the set $\{0\}$. But if $\Sigma \supset \Sigma_0$, $\mathcal{H}_\Sigma^+$ contains solutions of form (4.3.3), so there are no compact uniformly attracting sets.

Due to Remark 4.2.3, the analogues of all results of Section 4.2 (except the ones from Section 4.2.7) are true in the non-autonomous case. In particular, we have the following statements.

Theorem 4.3.1 (see [77]). *Assume that there exists a uniform trajectory semiattractor P for equation (4.3.1). Then there exists a minimal uniform trajectory attractor $\mathcal{U}$ for equation (4.3.1). Here one has $\Pi_+\mathcal{K}(\mathcal{H}_\Sigma^+) \subset \mathcal{U} = \Pi_+\mathcal{K}(\mathcal{U}) \subset \Pi_+\mathcal{K}(P)$ and the kernel $\mathcal{K}(\mathcal{H}_\Sigma^+)$ is relatively compact in $C((-\infty, +\infty); E_0)$ and bounded in $L_\infty(-\infty, +\infty; E)$.*

Theorem 4.3.2. *Assume that there exists a uniformly absorbing set P for equation (4.3.1) which is compact in $C([0, +\infty); E_0)$ and bounded in $L_\infty(0, +\infty; E)$. Then there exists a minimal uniform trajectory attractor $\mathcal{U}$ for equation (4.3.1). Here one has $\Pi_+\mathcal{K}(\mathcal{H}_\Sigma^+) \subset \mathcal{U} = \Pi_+\mathcal{K}(\mathcal{U}) \subset \Pi_+\mathcal{K}(P)$ and the kernel $\mathcal{K}(\mathcal{H}_\Sigma^+)$ is relatively compact in $C((-\infty, +\infty); E_0)$ and bounded in $L_\infty(-\infty, +\infty; E)$.*

Theorem 4.3.3 (see [77]). *If there exists a minimal uniform trajectory attractor $\mathfrak{U}$ for equation (4.3.1), then there is a uniform global attractor $\mathcal{A}$ for equation (4.3.1) and for all $t \geq 0$ one has*

$$\mathcal{K}(\mathcal{H}_\Sigma^+)(t) \subset \mathcal{A} = \mathfrak{U}(t) = \mathcal{K}(\mathfrak{U})(t).$$

Theorem 4.3.4. *Assume that there exists a uniformly attracting set P for equation (4.3.1) which is compact in $C([0, +\infty); E_0)$ and bounded in $L_\infty(0, +\infty; E)$. Then there exists a homogeneous uniform trajectory quasiattractor $\mathfrak{U} \subset P$ for equation (4.3.1) such that the set $\mathcal{A} = \mathfrak{U}(t)$, $t \geq 0$, is a uniform global attractor for equation (4.3.1).*

Lemma 4.3.1. *Let P be a uniformly attracting (or absorbing) set for equation (4.3.1) which is relatively compact in $C([0, +\infty); E_0)$ and bounded in $L_\infty(0, +\infty; E)$. Then its closure $\overline{P}$ in $C([0, +\infty); E_0)$ is a uniformly attracting (resp. absorbing) set for equation (4.3.1) which is compact in $C([0, +\infty); E_0)$ and bounded in $L_\infty(0, +\infty; E)$. If in addition $T(t)P \subset P$ for any $t \geq 0$, then $\overline{P}$ is a uniform trajectory semiattractor.*

Lemma 4.3.2. *Let $\mathfrak{U} \subset \mathcal{H}_\Sigma^+$ be a uniform trajectory quasiattractor. Then*

a) *$\mathfrak{U}$ is contained in any compact uniformly attracting set $P \subset C([0, +\infty); E_0)$;*

b) *in $\mathcal{H}_\Sigma^+$ there is no uniform trajectory quasiattractors different from $\mathfrak{U}$;*

c) *if, in addition, it is known that $T(h)\mathfrak{U} \subset \mathcal{H}_\Sigma^+$ for all $h \geq 0$, then $\mathfrak{U}$ is a minimal uniform trajectory attractor.*

Theorem 4.3.5. *Assume that there exists a uniformly attracting set P for equation (4.3.1) which is compact in $C([0, +\infty); E_0)$ and bounded in $L_\infty(0, +\infty; E)$. Let $T(h)P \subset \mathcal{H}_\Sigma^+$ for all $h \geq 0$. Then there exist a minimal uniform trajectory attractor $\mathfrak{U} = \Pi_+\mathcal{K}(\mathcal{H}_\Sigma^+)$ for equation (4.3.1) and a uniform global attractor $\mathcal{A} = \mathfrak{U}(t) = \mathcal{K}(\mathcal{H}_\Sigma^+)(t)$, $t \geq 0$ for equation (4.3.1).*

Remark 4.3.4. In [66] (Theorem 4.1, p. 210) this theorem was proved under additional assumptions that the family of trajectory spaces is translation-coordinated (see Remark 4.3.2) and that the symbol space Σ is a compact metric space.

Theorem 4.3.6. *Assume that there exists a minimal uniform trajectory attractor $\mathfrak{U}$ for equation (4.3.1). Then*

$$\Pi_+\mathcal{K}(\mathcal{H}_\Sigma^+) \subset \mathfrak{U} \subset \Pi_+\mathcal{K}\Big(\Big[\bigcup_{t\geq 0} T(t)\mathcal{H}_\Sigma^+\Big]\Big). \tag{4.3.9}$$

Assume now that Σ is a Hausdorff topological space, and the semigroup $T(t)$ acts on Σ.

Definition 4.3.9 (cf. [15]). The family of trajectory spaces $\{\mathcal{H}^+_\sigma\}$ is said to be *closed* if for any sequence of symbols $\sigma_m \to \sigma_0$ and for any sequence of trajectories $u_m \in \mathcal{H}^+_{\sigma_m}$ converging in $C([0,+\infty); E_0)$ and being bounded in $L_\infty(0,+\infty; E)$, one has $\lim\limits_{m\to+\infty} u_m \in \mathcal{H}^+_{\sigma_0}$.

Corollary 4.3.1. *Let the family of trajectory spaces $\{\mathcal{H}^+_\sigma\}$ be closed and translation-coordinated. Let Σ be compact. If there exists a minimal uniform trajectory attractor $\mathfrak{U}$ for equation (4.3.1), then*

$$\mathfrak{U} = \Pi_+ \mathcal{K}(\mathcal{H}^+_\Sigma). \tag{4.3.10}$$

Proof. Since $\{\mathcal{H}^+_\sigma\}$ is translation-coordinated, we have $T(h)\mathcal{H}^+_\Sigma \subset \mathcal{H}^+_\Sigma$ for all $h \geq 0$. Let us prove that $[\mathcal{H}^+_\Sigma] = \mathcal{H}^+_\Sigma$. Take any sequence of trajectories $u_m \in \mathcal{H}^+_{\sigma_m}$, $u_m \to u_0$ in $C([0,+\infty); E_0)$ that is bounded in $L_\infty(0,+\infty; E)$. Since Σ is compact, without loss of generality there is $\sigma_0 \in \Sigma$ such that $\sigma_m \to \sigma_0$. Since the family $\{\mathcal{H}^+_\sigma\}$ is closed, $u_0 \in \mathcal{H}^+_{\sigma_0} \subset \mathcal{H}^+_\Sigma$. Now (4.3.9) yields (4.3.10). □

Thus an analogue of Corollary 4.2.2 may be written as

Corollary 4.3.2. *Under the conditions of Theorem 4.3.4, let the family of trajectory spaces $\{\mathcal{H}^+_\sigma\}$ be closed and translation-coordinated and let Σ be compact. Then the homogeneous uniform trajectory quasiattractor $\mathfrak{U}$ is a minimal uniform trajectory attractor, and*

$$\mathfrak{U} = \Pi_+ \mathcal{K}(\mathcal{H}^+_\Sigma), \tag{4.3.11}$$

whereas for the uniform global attractor one has the expression

$$\mathcal{A} = \mathcal{K}(\mathcal{H}^+_\Sigma)(t) \tag{4.3.12}$$

for any $t \geq 0$.

Chapter 5

Strong solutions for equations of motion of viscoelastic medium

5.1 The Guillopé–Saut theorem

This section is concerned with the initial-boundary value problem describing the dynamics of a homogeneous incompressible viscoelastic medium in a bounded domain in $\mathbb{R}^n$, $n = 2, 3$. The constitutive relation which we consider here is Jeffreys' law (1.3.32) with Oldroyd's derivative (1.3.36). Combining it with the equation of motion (1.1.12), the incompressibility condition (1.1.10) and the no-slip condition (1.1.15), we get the following problem:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} - \mathrm{Div}\ \sigma + \mathrm{grad}\, p = f_0,\ (t, x) \in [0, T] \times \Omega, \tag{5.1.1}$$

$$\sigma + \lambda_1 \frac{D_{\mathbf{a}}\sigma}{Dt} = 2\eta\big(\mathcal{E}(u) + \lambda_2 \frac{D_{\mathbf{a}}\mathcal{E}(u)}{Dt}\big),\ (t, x) \in [0, T] \times \Omega, \tag{5.1.2}$$

$$\mathrm{div}\, u = 0,\ (t, x) \in [0, T] \times \Omega, \tag{5.1.3}$$

$$u(t, x) = 0,\ (t, x) \in [0, T] \times \partial\Omega. \tag{5.1.4}$$

Here u is the unknown velocity vector, σ is the unknown extra-stress tensor, p is the unknown scalar pressure function, f_0 is the given body force (all of them depend on a point x of a sufficiently regular bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ and on a moment of time t). The divergences div and Div and the gradient grad are taken with respect to the variable x. Besides, $\mathcal{E}(u) = (\mathcal{E}_{ij}(u))$, $\mathcal{E}_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the strain velocity tensor, $\eta > 0$ is the viscosity of the medium, λ_1 is the relaxation time, λ_2 is the retardation time, $0 < \lambda_2 < \lambda_1$, $-1 \le \mathbf{a} \le 1$.

Denote $\mu_1 = \eta\dfrac{\lambda_2}{\lambda_1}$, $\eta_1 = \eta - \mu_1$, $\tau = \sigma - 2\mu_1\mathcal{E}(u)$. Proceeding as in Section 1.5.1, and remembering (1.3.36), (1.3.7) and (1.3.34), we can rewrite (5.1.1) and (5.1.2) as follows:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} - \mu_1 \Delta u - \mathrm{Div}\ \tau + \mathrm{grad}\, p = f_0, \tag{5.1.5}$$

$$\tau + \lambda_1\big(\frac{\partial \tau}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial \tau}{\partial x_i} + \tau W - W\tau - \mathbf{a}(\tau\mathcal{E} + \mathcal{E}\tau)\big) = 2\eta_1\mathcal{E}(u). \tag{5.1.6}$$

Remember that here $W = (W_{ij}(u))$, $W_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i})$.

The main result of this section is

Theorem 5.1.1. *Given*

$$f_0 \in L_2(0,T;H^1(\Omega)^n) \cap W_2^1(0,T;H^{-1}(\Omega)^n),$$

and

$$a \in H^2(\Omega)^n \cap V,\ \tau_0 \in H^2_M(\Omega),$$

there exist $t_0 > 0$ *and a triple* (u, p, τ) *from the class*

$$\begin{aligned} u \in L_2(0,t_0;H^3(\Omega)^n) \cap C([0,t_0];H^2(\Omega)^n \cap V) \\ \cap W_2^1(0,t_0;V) \cap C^1([0,t_0];L_2(\Omega)^n), \end{aligned} \tag{5.1.7}$$

$$p \in L_2(0,t_0;H^2(\Omega)), \tag{5.1.8}$$

$$\tau \in C([0,t_0];H^2_M(\Omega)) \cap C^1([0,t_0];H^1_M(\Omega)) \tag{5.1.9}$$

which is a solution to problem (5.1.3) – (5.1.6) and satisfies the initial condition

$$u(0) = a, \quad \tau(0) = \tau_0. \tag{5.1.10}$$

Remark 5.1.1. The result is due to C. Guillopé and J.C. Saut [21], Theorem 2.4. R. Talhouk [60] generalized it onto the case of unbounded domains. Some other local existence and uniqueness results for problem (5.1.3) – (5.1.6) may be found in [13, 24, 25].

Remark 5.1.2. The uniqueness of u and τ in Theorem 5.1.1 can be proved by standard methods ([21], Theorem 2.5, [25], Theorem 3.1, see also the proof of uniqueness in Lemma 5.4.4 of the current chapter).

Remark 5.1.3. As in Theorem 3.2.1, an a priori estimate for $\|u(t)\|_2$ and $\|\tau(t)\|_2$ in Theorem 5.1.1 yields that $t_0 = T$. Such an a priori estimate can be proved provided the numbers η_1, $\|a\|_2$, $\|\tau_0\|_2$, $\|f_0\|_{L_2(0,T;H^1(\Omega)^n)}$ and $\|f_0'\|_{L_2(0,T;H^{-1}(\Omega)^n)}$ are small enough, see [21], Theorem 3.3, (or [25], for more general results of this type). In the case $\Omega = \mathbb{R}^n$ the global in time existence may be proved for arbitrary $\eta_1 > 0$ even for more general models (see below, Section 5.2). Let us mention also paper [13] with global existence results for small a, τ_0, f_0 in Besov spaces (Ω is $\mathbb{R}^n$ or torus).

Remark 5.1.4. If f is small and independent of time, one can prove also the existence of time-independent (*stationary*) solutions for problem (5.1.3) – (5.1.6) (see [50, 51, 21, 22, 25]).

Remark 5.1.5. Conditions (5.1.3), (5.1.4) are understood in the sense

$$u(t) \in V,\ t \in [0,T]$$

(cf. Section 2.1.2).

The proof of Theorem 5.1.1 requires some lemmas on solvability of linear Stokes' and transport problems.

Consider the Stokes problem in the form

$$\frac{\partial u}{\partial t} - \mu_1 \Delta u + \operatorname{grad} p = f, \quad (t,x) \in [0,T] \times \Omega, \tag{5.1.11}$$

$$\operatorname{div} u = 0, \quad (t,x) \in [0,T] \times \Omega, \tag{5.1.12}$$

$$u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \tag{5.1.13}$$

$$u(0,x) = a(x), \quad x \in \Omega. \tag{5.1.14}$$

Lemma 5.1.1 (see [61], p. 181). *Given $f \in L_2(0,T; L_2(\Omega)^n)$ and $a \in V$, there exists a solution (u,p) to problem (5.1.11) – (5.1.14) from the class*

$$u \in L_2(0,T; H^2(\Omega)^n) \cap C([0,T]; V) \cap W_2^1(0,T; H), \tag{5.1.15}$$

$$p \in L_2(0,T; H^1(\Omega)). \tag{5.1.16}$$

The pair $(u, \operatorname{grad} p)$ is unique. The solution (u,p) is arbitrary regular with respect to t and x is provided Ω, f and a are regular enough.

Denote by Ω^k, $k = 1, 2, \dots$, the connected components of Ω. Fix a function $\vartheta \in L_2(\Omega)$ with $\int_{\Omega^k} \vartheta(x)\, dx \neq 0$ for each k. Denote by $E^m_{\vartheta,\Omega}$ the subspace of $H^m(\Omega)$, $m = 1, 2$, which is the intersection of the kernels of the functionals $(\cdot, \vartheta)_{L_2(\Omega^k)}$. Lemma 5.1.1 and Corollary 3.1.3 imply

Corollary 5.1.1. *The operator*

$$\begin{aligned} S : {} & L_2(0,T; H^2(\Omega)^n) \cap C([0,T]; V) \cap W_2^1(0,T; H) \times L_2(0,T; E^1_{\vartheta,\Omega}) \\ & \to L_2(0,T; L_2(\Omega)^n) \times V, \\ S(u,p) = {} & (\frac{\partial u}{\partial t} - \mu_1 \Delta u + \operatorname{grad} p, u(0)), \end{aligned}$$

is an isomorphism.

Corollary 5.1.2. S^{-1} *is a bounded operator from*

$$L_2(0,T; H^1(\Omega)^n) \cap C([0,T]; L_2(\Omega)^n) \cap W_2^1(0,T; H^{-1}(\Omega)^n) \times H^2(\Omega)^n \cap V$$

into

$$L_2(0,T;H^3(\Omega)^n)\cap C([0,T];H^2(\Omega)^n\cap V)\cap W_2^1(0,T;V)\cap C^1([0,T];H)$$
$$\times L_2(0,T;E^2_{\vartheta,\Omega}).$$

Proof. If f and a are more regular with respect to t and x, $S^{-1}(f,a)$ is also more regular. Since the smooth functions are dense in $L_2(0,T;H^1(\Omega)^n)\cap C([0,T];L_2(\Omega)^n)\cap W_2^1(0,T;H^{-1}(\Omega)^n)\times H^2(\Omega)^n\cap V$, it suffices to show that S^{-1} is bounded on the sufficiently regular functions.

Denote $S^{-1}(f,a)$ by (u,p), where (f,a) are regular enough. We have:

$$\frac{\partial u}{\partial t}-\mu_1\Delta u+\operatorname{grad}p=f. \tag{5.1.17}$$

This implies

$$u'(0)-\mu_1\Delta a+\operatorname{grad}p(0)=f(0),$$

so

$$\|u'(0)+\operatorname{grad}p(0)\|\le\mu_1\|\Delta a\|+\|f(0)\|.$$

But u' and grad p are orthogonal in $L_2(\Omega)^n$ (this follows from Lemma 3.1.1 or direct integration by parts using the condition $\operatorname{div}u'=0$). Thus,

$$\|u'(0)\|\le\mu_1\|a\|_2+\|f\|_{C([0,T];L_2(\Omega)^n)}. \tag{5.1.18}$$

Differentiate (5.1.17) with respect to t:

$$\frac{\partial u'}{\partial t}-\mu_1\Delta u'+\frac{\partial}{\partial t}\operatorname{grad}p=f'.$$

Taking the scalar product of each term with u' in $L_2(\Omega)^n$ for each $t\in[0,T]$, we get:

$$\frac{d}{dt}(u'(t),u'(t))-\mu_1\int_\Omega\Delta u'(t)(x)\,u'(t)(x)\,dx+\int_\Omega\frac{\partial\operatorname{grad}p(t)(x)}{\partial t}u'(t)(x)\,dx$$
$$=(f'(t),u'(t)).$$

Integrating by parts (cf. Section 6.1.2) in the second and the third terms and taking into account that $\operatorname{div}u'=0$, we obtain:

$$\frac{d}{dt}(u',u')+\mu_1(\nabla u',\nabla u')=(f',u').$$

Integration from 0 to t yields

$$\|u'(t)\|^2+\mu_1\int_0^t\|\nabla u'(s)\|^2ds\le\|u'(0)\|^2+\int_0^t\|f'(s)\|_{-1}\|u'(s)\|_1\,ds,$$

so by inequality (2.1.28)

$$\|u'(t)\|^2 + \mu_1 \int_0^t \|\nabla u'(s)\|^2 ds \\ \le \|u'(0)\|^2 + (1 + K_0^2(\Omega))^{1/2} \int_0^t \|f'(s)\|_{-1} \|\nabla u'(s)\| \, ds.$$

Application of Cauchy's inequality $ab \le \frac{a^2}{2c} + \frac{b^2 c}{2}$ to the last integral implies

$$\|u'(t)\|^2 + \mu_1 \int_0^t \|\nabla u'(s)\|^2 ds \\ \le \|u'(0)\|^2 + (1 + K_0^2(\Omega))^{1/2} \int_0^t \Big(\frac{(1 + K_0^2(\Omega))^{1/2}}{2\mu_1} \|f'(s)\|_{-1}^2 \\ + \frac{\mu_1}{2(1 + K_0^2(\Omega))^{1/2}} \|\nabla u'(s)\|^2 \Big) \, ds,$$

so

$$\|u'(t)\|^2 + \frac{\mu_1}{2} \int_0^t \|\nabla u'(s)\|^2 ds \le \|u'(0)\|^2 + \frac{(1 + K_0^2(\Omega))}{2\mu_1} \int_0^t \|f'(s)\|_{-1}^2 \, ds. \tag{5.1.19}$$

Now, if we take a pair (f, a) from a fixed bounded set in $L_2(0, T; H^1(\Omega)^n) \cap C([0, T]; L_2(\Omega)^n) \cap W_2^1(0, T; H^{-1}(\Omega)^n) \times H^2(\Omega)^n \cap V$, then, by estimates (5.1.18) and (5.1.19), u' is bounded in $L_2(0, T; H^1(\Omega)^n) \cap C([0, T]; L_2(\Omega)^n)$ by a constant independent of (f, a). But $-\mu_1 \Delta u + \operatorname{grad} p = f - u'$, so, by a classical regularity result for the stationary Stokes problem ([61], Proposition 1.2.2), u is bounded in $L_2(0, T; H^3(\Omega)^n) \cap C([0, T]; H^2(\Omega)^n)$ and p is bounded in $L_2(0, T; H^2(\Omega))$. The proof is complete. □

Now, let us give an existence result for the following equation of transport type:

$$\tau + \lambda_1 \Big(\frac{\partial \tau}{\partial t} + \sum_{i=1}^n v_i \frac{\partial \tau}{\partial x_i} + \tau W(v) - W(v)\tau - \mathbf{a}(\tau \mathcal{E}(v) + \mathcal{E}(v)\tau) \Big) \\ = 2\eta_1 \mathcal{E}(v), \ (t, x) \in [0, T] \times \Omega. \tag{5.1.20}$$

Here $v(t, x)$ is a given velocity field, and τ is unknown.

Lemma 5.1.2 (see [21], Lemma 2.3). *Given $v \in L_1(0, T; H^3(\Omega)^n) \cap L_\infty(0, T; H^2(\Omega)^n \cap V)$, $\tau_0 \in H_M^2(\Omega)$, there exists a unique (e.g. in the class $C([0, T]; H_M^1(\Omega))$) solution $\tau \in C([0, T]; H_M^2(\Omega)) \cap W_\infty^1(0, T; H_M^1(\Omega))$ to equation (5.1.20), satisfying the initial condition*

$$\tau(0) = \tau_0. \tag{5.1.21}$$

One has the following estimate for this solution:

$$\|\tau\|_{C([0,T];H^2_M(\Omega))} \leq (\|\tau_0\|_2 + K_1)\exp(K_2\|v\|_{L_1(0,T;H^3(\Omega)^n)}) \tag{5.1.22}$$

$$\begin{aligned}\|\tau'\|&_{L_\infty(0,T;H^1_M(\Omega))} \\ &\leq K_3(\|v\|_{L_\infty(0,T;H^2(\Omega)^n)} + K_4)(\|\tau_0\|_2 + K_1) \\ &\quad \cdot \exp(K_2\|v\|_{L_1(0,T;H^3(\Omega)^n)}).\end{aligned} \tag{5.1.23}$$

Here $K_1, \dots, K_4$ depend only on Ω, λ_1, η_1.

If, in addition, $v \in C([0,T]; H^2(\Omega)^n \cap V)$, then $\tau \in C^1([0,T]; H^1_M(\Omega))$.

Now we are ready to turn to

Proof of Theorem 5.1.1. Consider the set $Q = Q(t_0, R)$ which consists of the pairs

$$\begin{aligned}(v, \varsigma); \quad v &\in L_2(0,t_0; H^3(\Omega)^n) \cap L_\infty(0,t_0; H^2(\Omega)^n \cap V) \\ &\cap C_w([0,t_0]; H^2(\Omega)^n) \cap W^1_2(0,t_0; V) \cap W^1_\infty(0,t_0; L_2(\Omega)^n); \\ \varsigma &\in L_\infty(0,t_0; H^2_M(\Omega)) \cap C_w([0,t_0]; H^2_M(\Omega)) \cap W^1_\infty(0,t_0; H^1_M(\Omega));\end{aligned}$$

$$\begin{aligned}\|v\|^2_{L_2(0,t_0;H^3(\Omega)^n)} + \|v\|^2_{L_\infty(0,t_0;H^2(\Omega)^n)} + \|v\|^2_{W^1_2(0,t_0;V)} + \|v\|^2_{W^1_\infty(0,t_0;L_2(\Omega)^n)} \\ + \|\varsigma\|_{L_\infty(0,t_0;H^2_M(\Omega))} + \|\varsigma\|_{W^1_\infty(0,t_0;H^1_M(\Omega))} \leq R; \\ v(0) = a;\ \varsigma(0) = \tau_0\end{aligned}$$

(the parameters $t_0, R > 0$ will be defined below). This set is convex and non-empty for R large enough (it contains, for example, the pair (u_*, τ_*) where $(u_*, p_*) = S^{-1}(0,a)$ and $\tau_*(t) \equiv \tau_0$). By Theorem 2.2.6 it is relatively compact in the space $X = C([0,t_0]; V) \times C([0,t_0]; H^1_M(\Omega))$. Moreover, it is closed in X. Really, for any sequence $(v_m, \varsigma_m) \in Q$ converging to a pair (v_0, ς_0) in the topology of X, one has $v_0(0) = a$, $\varsigma_0(0) = \tau_0$; $v_m \rightharpoonup v_0$ weakly in $L_2(0,t_0; H^3(\Omega)^n)$ and in $W^1_2(0,t_0; V)$, $*$-weakly in $L_\infty(0,t_0; H^2(\Omega)^n)$ and in $W^1_\infty(0,t_0; L_2(\Omega)^n)$; $\varsigma_m \rightharpoonup \varsigma_0$ $*$-weakly in $L_\infty(0,t_0; H^2_M(\Omega))$ and in $W^1_\infty(0,t_0; H^1_M(\Omega))$. Hence, v_0 belongs to $L_\infty(0,t_0; H^2(\Omega)^n)$ and to $W^1_2(0,t_0; V) \subset C_w([0,t_0]; V)$, so, by Lemma 2.2.6, $v_0 \in C_w([0,t_0]; H^2(\Omega)^n)$. Analogously, $\varsigma_0 \in C_w([0,t_0]; H^2_M(\Omega))$. Thus, $(v_0, \varsigma_0) \in Q$.

Consider the mapping:

$$\phi : Q \subset X \to X,$$

$$\phi(v, \varsigma) = (u, \tau),$$

u is the first component of the solution of Stokes' problem $(u,p) = [S^{-1}(f,a)]\big|_{[0,t_0]}$ where $f = f_0 + \operatorname{Div}\varsigma - \sum_{i=1}^n v_i \frac{\partial v}{\partial x_i}$; τ is the solution of the transport problem (5.1.20) – (5.1.21), restricted to the segment $[0,t_0]$. Observe that if (v, ς) is a fixed point of ϕ, then the corresponding triple (u, p, τ) is a solution to (5.1.3) – (5.1.6) in class (5.1.7) – (5.1.9).

Let us show that $\phi(Q) \subset Q$ for appropriate t_0 and R. Below in the proof C stands for various constants which do not depend on t_0 and R.

Take a pair $(v, \varsigma) \in Q$ and let $(u, \tau) = \phi(v, \varsigma)$.

Estimate (2.1.21) gives $\| \sum\limits_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} \|_1 \le C \|v\|_2^2$.

Using Lemma 2.2.1, a), we obtain:

$$
\begin{aligned}
&\|f\|_{L_2(0,t_0;H^1(\Omega)^n)} \\
&\quad \le \|f_0\|_{L_2(0,t_0;H^1(\Omega)^n)} + \|\operatorname{Div} \varsigma\|_{L_2(0,t_0;H^1(\Omega)^n)} \\
&\qquad + \Big\| \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} \Big\|_{L_2(0,t_0;H^1(\Omega)^n)} \\
&\quad \le \|f_0\|_{L_2(0,t_0;H^1(\Omega)^n)} + t_0^{1/2} \|\operatorname{Div} \varsigma\|_{L_\infty(0,t_0;H^1(\Omega)^n)} \\
&\qquad + t_0^{1/2} \Big\| \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} \Big\|_{L_\infty(0,t_0;H^1(\Omega)^n)} \\
&\quad \le \|f_0\|_{L_2(0,t_0;H^1(\Omega)^n)} + C t_0^{1/2} \|\varsigma\|_{L_\infty(0,t_0;H^2_M(\Omega))} \\
&\qquad + C t_0^{1/2} \|v\|^2_{L_\infty(0,t_0;H^2(\Omega)^n)} \\
&\quad \le C + C R t_0^{1/2}.
\end{aligned}
\tag{5.1.24}
$$

Take a function $\varphi \in H^1(\Omega)^n$. Applying integration by parts and estimate (2.1.19), and remembering that $\operatorname{div} v = 0$, we get

$$
\begin{aligned}
\Big|\Big(\sum_{i=1}^{n} \frac{\partial}{\partial t}\Big(v_i(t) \frac{\partial v(t)}{\partial x_i}\Big), \varphi\Big)\Big| &= \Big|\Big(\sum_{i=1}^{n} v_i'(t) \frac{\partial v(t)}{\partial x_i}, \varphi\Big) + \Big(\sum_{i=1}^{n} v_i(t) \frac{\partial v'(t)}{\partial x_i}, \varphi\Big)\Big| \\
&= \Big|\Big(\sum_{i=1}^{n} v_i'(t) v(t), \frac{\partial \varphi}{\partial x_i}\Big) + \Big(\sum_{i=1}^{n} v_i(t) v'(t), \frac{\partial \varphi}{\partial x_i}\Big)\Big| \\
&\le C \|v'\|_0 \|v\|_2 \|\varphi\|_1,
\end{aligned}
$$

so

$$
\Big\| \sum_{i=1}^{n} \frac{\partial}{\partial t}\Big(v_i \frac{\partial v}{\partial x_i}\Big) \Big\|_{-1}(t) \le C \|v'(t)\|_0 \|v(t)\|_2.
$$

Thus we have

$$
\begin{aligned}
&\|f'\|_{L_2(0,t_0;H^{-1}(\Omega)^n)}\\
&\quad\le \|f_0'\|_{L_2(0,t_0;H^{-1}(\Omega)^n)} + \|\operatorname{Div}\varsigma'\|_{L_2(0,t_0;H^{-1}(\Omega)^n)}\\
&\qquad + \Big\|\sum_{i=1}^n \frac{\partial}{\partial t}\Big(v_i\frac{\partial v}{\partial x_i}\Big)\Big\|_{L_2(0,t_0;H^{-1}(\Omega)^n)}\\
&\quad\le \|f_0'\|_{L_2(0,t_0;H^{-1}(\Omega)^n)} + t_0^{1/2}\|\operatorname{Div}\varsigma'\|_{L_\infty(0,t_0;H^{-1}(\Omega)^n)}\\
&\qquad + t_0^{1/2}\Big\|\sum_{i=1}^n \frac{\partial}{\partial t}\Big(v_i\frac{\partial v}{\partial x_i}\Big)\Big\|_{L_\infty(0,t_0;H^{-1}(\Omega)^n)}\\
&\quad\le \|f_0\|_{W_2^1(0,t_0;H^{-1}(\Omega)^n)} + Ct_0^{1/2}\|\varsigma\|_{W_\infty^1(0,t_0;H_M^1(\Omega))}\\
&\qquad + Ct_0^{1/2}\|v\|_{L_\infty(0,t_0;H^2(\Omega)^n)}\|v\|_{W_\infty^1(0,t_0;H)}\\
&\quad\le C + CRt_0^{1/2}.
\end{aligned}
\tag{5.1.25}
$$

By Lemma 2.2.7, $f_0, f \in C([0,T];L_2(\Omega)^n)$. Then, Lemma 2.2.7 and the Newton–Leibnitz formula give:

$$
\begin{aligned}
\max_{t\in[0,t_0]}\|f(t)\|_0^2 &\le \|f(0)\|_0^2 + \Big|2\int_0^{t_0}\langle f'(s), f(s)\rangle\,ds\Big|\\
&\le C\Big(\|f_0(0)\|_0^2 + \|\operatorname{Div}\tau_0\|_0^2 + \Big\|\sum_{i=1}^n a_i\frac{\partial a}{\partial x_i}\Big\|_0^2\\
&\qquad + \|f'\|_{L_2(0,t_0;H^{-1}(\Omega)^n)}\|f\|_{L_2(0,t_0;H^1(\Omega)^n)}\Big)\\
&\le C\big(\|f_0\|^2_{C([0,T];L_2(\Omega)^n)} + \|\tau_0\|_1^2 + \|a\|_2^4\\
&\qquad + \|f'\|_{L_2(0,t_0;H^{-1}(\Omega)^n)}\|f\|_{L_2(0,t_0;H^1(\Omega)^n)}\big)\\
&\le C(C + (C + CRt_0^{1/2})^2) \le (C + CRt_0^{1/2})^2.
\end{aligned}
\tag{5.1.26}
$$

Estimates (5.1.24) – (5.1.26) and Corollary 5.1.2 imply that

$$
\begin{aligned}
\|u\|^2_{L_2(0,t_0;H^3(\Omega)^n)} + \|u\|^2_{L_\infty(0,t_0;H^2(\Omega)^n)} + \|u\|^2_{W_2^1(0,t_0;V)} + \|u\|^2_{W_\infty^1(0,t_0;L_2(\Omega)^n)}&\\
\le (C + CRt_0^{1/2})^2.&
\end{aligned}
\tag{5.1.27}
$$

By Lemma 2.2.1, a), $\|v\|_{L_1(0,t_0;H^3(\Omega)^n)} \le t_0^{1/2}\|v\|_{L_2(0,t_0;H^3(\Omega)^n)} \le (Rt_0)^{1/2}$. Applying estimates (5.1.22) and (5.1.23), we get:

$$
\|\tau\|_{C([0,t_0];H_M^2(\Omega))} \le C\exp(CR^{1/2}t_0^{1/2}) \tag{5.1.28}
$$

$$
\|\tau'\|_{L_\infty(0,t_0;H_M^1(\Omega))} \le C(R^{1/2} + C)\exp(CR^{1/2}t_0^{1/2}). \tag{5.1.29}
$$

Estimates (5.1.27) – (5.1.29) give that, for sufficiently large R and sufficiently small t_0 (depending on R), one has $(u, \tau) \in Q$.

Let us prove now that ϕ is continuous. Take a sequence $(v_m, \varsigma_m) \in Q$ converging to a pair (v_0, ς_0) in the topology of X. Let $(u_m, \tau_m) = \phi(v_m, \varsigma_m)$ and $f^m = f_0 + \operatorname{Div} \varsigma_m - \sum_{i=1}^{n} v_{mi} \frac{\partial v_m}{\partial x_i}$. Then

$$
\begin{aligned}
&\| f^m - f^0 \|_{L_2(0,t_0;L_2(\Omega)^n)} \\
&\quad \le \| \operatorname{Div}(\varsigma_m - \varsigma_0) \|_{L_2(0,t_0;L_2(\Omega)^n)} + \| \sum_{i=1}^{n} (v_{mi} \frac{\partial v_m}{\partial x_i} - v_{0i} \frac{\partial v_0}{\partial x_i}) \|_{L_2(0,t_0;L_2(\Omega)^n)} \\
&\quad \le \| \varsigma_m - \varsigma_0 \|_{L_2(0,t_0;H^1_M(\Omega))} + \| \sum_{i=1}^{n} v_{mi} \frac{\partial (v_m - v_0)}{\partial x_i} \|_{L_2(0,t_0;L_2(\Omega)^n)} \\
&\qquad + \| \sum_{i=1}^{n} (v_{mi} - v_{0i}) \frac{\partial v_0}{\partial x_i} \|_{L_2(0,t_0;L_2(\Omega)^n)} \\
&\quad \le t_0^{1/2} \big(\| \varsigma_m - \varsigma_0 \|_{C([0,t_0];H^1_M(\Omega))} + \| \sum_{i=1}^{n} v_{mi} \frac{\partial (v_m - v_0)}{\partial x_i} \|_{L_\infty(0,t_0;L_2(\Omega)^n)} \\
&\qquad + \| \sum_{i=1}^{n} (v_{mi} - v_{0i}) \frac{\partial v_0}{\partial x_i} \|_{L_\infty(0,t_0;L_2(\Omega)^n)} \big) \\
&\quad \le t_0^{1/2} \big(\| \varsigma_m - \varsigma_0 \|_{C([0,t_0];H^1_M(\Omega))} \\
&\qquad + \| v_m \|_{L_\infty(0,t_0;H^2(\Omega)^n)} \| v_m - v_0 \|_{C([0,t_0];H^1(\Omega)^n)} \\
&\qquad + \| v_m - v_0 \|_{C([0,t_0];H^1(\Omega)^n)} \| v_0 \|_{L_\infty(0,t_0;H^2(\Omega)^n)} \big) \\
&\quad \le t_0^{1/2} \big(\| \varsigma_m - \varsigma_0 \|_{C([0,t_0];H^1_M(\Omega))} + R^{1/2} \| v_m - v_0 \|_{C([0,t_0];H^1(\Omega)^n)} \\
&\qquad + R^{1/2} \| v_m - v_0 \|_{C([0,t_0];H^1(\Omega)^n)} \big) \xrightarrow[m \to \infty]{} 0.
\end{aligned}
$$

Thus, the map $(v, \varsigma) \mapsto f$ is continuous from X to $L_2(0, t_0; L_2(\Omega)^n)$. By Corollary 5.1.1, the map $f \mapsto u$ is continuous from $L_2(0, t_0; L_2(\Omega)^n)$ to $C([0, t_0]; V)$. Hence, $u_m \to u_0$ in $C([0, t_0]; V)$.

Further, the inclusion $\phi(Q) \subset Q$ implies that the sequence $\{\tau_m\}$ is relatively compact in $C([0, t_0]; H^1_M(\Omega))$. Any accumulation point of this sequence is a solution to (5.1.20) – (5.1.21) with $v = v_0$. In fact, if $\tau_{m_k} \xrightarrow[k \to \infty]{} \tau_*$ in $C([0, t_0]; H^1_M(\Omega))$, then we can pass to the limit in the equality

$$
\begin{aligned}
&\tau_{m_k} + \lambda_1 \Big(\frac{\partial \tau_{m_k}}{\partial t} + \sum_{i=1}^{n} v_{m_k i} \frac{\partial \tau_{m_k}}{\partial x_i} + \tau_{m_k} W(v_{m_k}) - W(v_{m_k}) \tau_{m_k} - \\
&\qquad \mathbf{a} (\tau_{m_k} \mathcal{E}(v_{m_k}) + \mathcal{E}(v_{m_k}) \tau_{m_k}) \Big) = 2\eta_1 \mathcal{E}(v_{m_k})
\end{aligned}
\tag{5.1.30}
$$

at least in the sense of distributions.

But, since τ_0 is the only solution of (5.1.20) – (5.1.21) with $v = v_0$, we conclude that $\tau_m \to \tau_0$ in $C([0,t_0]; H^1_M(\Omega))$.

Thus, the conditions of Theorem 3.2.4 are fulfilled for ϕ, so ϕ has a required fixed point. □

5.2 Initial-value problem for the combined model of nonlinear viscoelastic medium

5.2.1 Formulation of the initial-value problem

In the remainder of the chapter we investigate the initial-value problem describing the motion of homogeneous incompressible nonlinear viscoelastic medium with the combined constitutive law (1.5.15) – (1.5.19) described in Section 1.5.2. We shall consider the case of motion in the whole space $\mathbb{R}^n$. Combining the constitutive relations with the equation of motion (1.1.9) and the incompressibility condition (1.1.10), we get the following problem:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} = \mathrm{Div}\ \mathrm{T} + f_0, \quad (t,x) \in [0,T] \times \mathbb{R}^n, \tag{5.2.1}$$

$$\operatorname{div} u = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^n, \tag{5.2.2}$$

$$\mathrm{T} = -pI + \sum_{k=0}^{r} \tau^k, \tag{5.2.3}$$

$$\tau^k + \lambda_k \frac{D_0 \tau^k}{Dt} + \beta_k(\tau^k, \mathcal{E}) = 2\eta_k \mathcal{E}(u), \quad k = 1, \dots, r, \tag{5.2.4}$$

$$\tau^0 = \Psi(\mathcal{E}) = \varphi_1 \mathcal{E} + \varphi_2 \mathcal{E}^2, \tag{5.2.5}$$

$$\begin{aligned} \beta_k(\tau, \mathcal{E}) = \alpha_0^k I + \alpha_1^k \mathcal{E} + \alpha_2^k \mathcal{E}^2 + \alpha_3^k \tau + \alpha_4^k \tau^2 + \alpha_5^k (\mathcal{E}\tau + \tau\mathcal{E}) + \\ + \alpha_6^k (\mathcal{E}^2 \tau + \tau \mathcal{E}^2) + \alpha_7^k (\mathcal{E}\tau^2 + \tau^2 \mathcal{E}) + \alpha_8^k (\mathcal{E}^2 \tau^2 + \tau^2 \mathcal{E}^2). \end{aligned} \tag{5.2.6}$$

Here u is the unknown velocity vector, T is the unknown stress tensor, f_0 is the given body force (all of them depend on a point x of the space $\mathbb{R}^n$, $n = 2, 3$, and on a moment of time t); $\lambda_k > 0$ are relaxation times, $\eta_k > 0$ are viscosities; φ_1, φ_2 and α_j^k are scalar functions:

$$\varphi_i = \varphi_i(\mathrm{Tr}(\mathcal{E}^2), \det \mathcal{E}), \quad i = 1, 2, \tag{5.2.7}$$

$$\begin{aligned} \alpha_j^k = \alpha_j^k \big(\mathrm{Tr}\, \mathcal{E}^2, \mathrm{Tr}\, \mathcal{E}^3, \mathrm{Tr}(\tau), \mathrm{Tr}(\tau^2), \mathrm{Tr}(\tau^3), \mathrm{Tr}(\tau\mathcal{E}), \mathrm{Tr}(\tau^2\mathcal{E}), \\ \mathrm{Tr}(\tau\mathcal{E}^2), \mathrm{Tr}(\tau^2\mathcal{E}^2) \big), \quad k = 1, \dots, r; \ j = 0, \dots, 8. \end{aligned} \tag{5.2.8}$$

Remark 5.2.1. For convenience, we omitted tildes over p and φ_1 (cf. Remark 1.4.4).

Generally, the pressure $p(t, x)$ may be determined up to an arbitrary scalar function of time (cf. Lemma 5.1.1). To provide uniqueness, fix a function $\vartheta \in L_2(\mathbb{R}^n)$ with a compact support and satisfying $\int_{\mathbb{R}^n} \vartheta(x)\, dx \neq 0$, and assume that

$$\int_{\mathbb{R}^n} p(t, x)\vartheta(x)dx \equiv 0. \tag{5.2.9}$$

The initial conditions have the following form:

$$u(0, x) = a(x),\ \tau^k(0, x) = \tau_0^k(x), \quad x \in \mathbb{R}^n, \quad k = 1, \dots, r, \tag{5.2.10}$$

where a and τ_0^k are prescribed functions.

5.2.2 The Leray projection in $\mathbb{R}^n$ and some additional notations

Below in this chapter, in addition to the notations already introduced in the book, we use also the following notations.

We shall use the function spaces of Sobolev–Slobodetskii type

$$H_V^s = \{u \in H^s(\mathbb{R}^n, \mathbb{R}^n),\ \operatorname{div} u = 0\}$$

and

$$H_M^s = H_M^s(\mathbb{R}^n) = H^s(\mathbb{R}^n, \mathbb{R}_S^{n\times n}),$$

where $s \in \mathbb{R}, s \geq 0$.

A scalar product in these spaces can be given by the equality

$$(u, v)_s = (B_0^{\frac{s}{2}} u, B_0^{\frac{s}{2}} v)_0 \tag{5.2.11}$$

where B_0 is formally defined as $I - \Delta$, identity operator minus Laplacian. The operators $B_0 : H_V^2 \subset H_V^0 \to H_V^0$ and $B_0 : H_M^2 \subset H_M^0 \to H_M^0$ are self-adjoint and strongly positive. The powers of such operators were defined in Section 3.1.4.

Remark 5.2.2. In the case of natural s the notation $(u, v)_s$ was already introduced in Section 2.1.1:

$$(u, v)_s = \sum_{|\alpha|\leq s} (D^\alpha u, D^\alpha v)_0.$$

However, it is easy to observe that this expression coincides with (5.2.11) for natural s.

The Euclidean norm in both spaces will be denoted as $\|\cdot\|_s$. This norm is equivalent to the Sobolev–Slobodetskii norm introduced in Section 2.1.1 (see e.g. [34]).

Now we are going to give a characterization of the space H^s_V. For this purpose, consider the *Leray projection* P, which is formally defined as:

$$(Pu)_i = \sum_{j=1}^{n} \mathcal{F}^{-1}(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2})\mathcal{F}(u_j), \tag{5.2.12}$$

where $\mathcal{F}_{x\to\xi}$ is the Fourier transform of $\mathbb{R}^n$, and δ_{ij} is the Kronecker delta, $i, j = 1, \dots, n$.

Sometimes (5.2.12) is written as

$$(Pu)_i = \sum_{j=1}^{n} \big(\delta_{ij} - \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Delta^{-1}\big)u_j$$

or

$$Pu = u - \text{grad div}\, \Delta^{-1} u.$$

It is easy to see that $P^2 = P$ and P is a continuous automorphism of $H^s(\mathbb{R}^n, \mathbb{R}^n)$ (we may assume that the scalar product in this space is given by (5.2.11)). Furthermore,

$$P(H^s) = H^s_V.$$

Really, (5.2.12) implies that for $u \in H^s$ one has

$$\text{div}\, Pu \equiv 0. \tag{5.2.13}$$

This implies the inclusion $P(H^s) \subset H^s_V$. But $P|_{H^s_V} = I$, so $H^s_V \subset P(H^s)$.

The operators P and B_0 commute. Therefore [84] P and the fractional powers of B_0 also commute.

Let us show that the norm of the operator P in H^s does not exceed 1. It suffices to show that

$$(Pu, Pu)_s \le (u, u)_s. \tag{5.2.14}$$

Since P is continuous, it is enough to prove (5.2.14) for smooth u.

We have:

$$(u, u)_s = (Pu, Pu)_s + 2((I - P)u, Pu)_s + ((I - P)u, (I - P)u)_s. \tag{5.2.15}$$

But

$$\begin{aligned}
((I - P)u, Pu)_s &= \int_{\mathbb{R}^n} B_0^{\frac{s}{2}}(I - P)u(x) B_0^{\frac{s}{2}} Pu(x) dx \\
&= \int_{\mathbb{R}^n} (I - P) B_0^{\frac{s}{2}} u(x) P B_0^{\frac{s}{2}} u(x) dx \\
&= \int_{\mathbb{R}^n} \text{grad div}\, \Delta^{-1} B_0^{\frac{s}{2}} u(x) P B_0^{\frac{s}{2}} u(x) dx \\
&= -\int_{\mathbb{R}^n} \text{div}\, \Delta^{-1} B_0^{\frac{s}{2}} u(x) \cdot \text{div}\, P B_0^{\frac{s}{2}} u(x) dx = 0
\end{aligned}$$

(we have integrated by parts and used (5.2.13)). So (5.2.15) yields

$$(u,u)_s = (Pu,Pu)_s + ((I-P)u,(I-P)u)_s \geq (Pu,Pu)_s.$$

Thus, $\|P\| \leq 1$. But $P^2 = P$. So [84] P is an orthogonal projector in H^s.

In addition to the introduced notations, we shall use the following ones. The symbol ∇_ξ stands for the Fréchet derivative of functions or matrix functions of one or two matrix arguments $\varphi : \mathbb{R}_S^{n\times n} \to \mathbb{R}$, $\beta : \mathbb{R}_S^{n\times n} \times \mathbb{R}_S^{n\times n} \to \mathbb{R}_S^{n\times n}$ etc. The partial derivative of a function $\varphi(\xi)$ of matrix argument $\xi = (\xi_{ij})$ with respect to an element ξ_{ij} will be denoted as $\frac{\partial\varphi}{\partial\xi_{ij}}$.

5.2.3 The main existence and uniqueness theorem

Let us turn now to the formulation of the existence and uniqueness theorem for a solution of problem (5.2.1) – (5.2.10).

Let $\eta_0 = \frac{\varphi_1(0,0)}{2}$. Assume that $\eta_0 > 0$. This is a natural condition since the physical meaning of η_0 is a viscosity parameter.

Let us also assume that φ_i and α_j^k are C^4- and C^3-smooth functions, respectively, and

$$\alpha_0^k(\theta) = \alpha_1^k(\theta) = \alpha_3^k(\theta) = 0, \quad \frac{\partial\alpha_0^k(\theta)}{\partial \mathrm{Tr}(\tau)} = 0$$

(θ stands for the point $(0,0,0,0,0,0,0,0,0)$). This assumption is also natural in the considered model since the functions β_k in it correspond to "nonlinear" effects, i.e. effects of the second order and higher. Therefore the coefficients α_1^k and α_3^k at the first order terms $\mathcal{E}$ and τ should be "of the first order", i.e. they should vanish in the point θ, and the coefficient α_0^k at the zero order term I should be of the second order, i.e. it should vanish in θ and its partial derivative with respect to the "linear" argument $\mathrm{Tr}(\tau)$ should vanish in θ.

Remark 5.2.3. Instead of the C^4- smoothness of φ_1 and φ_2 it is sufficient for these functions to be differentiable in zero provided the function $\Psi : \mathbb{R}_S^{n\times n} \to \mathbb{R}_S^{n\times n}$ is C^4 - smooth or has locally Lipschitz third derivatives (see Section 5.4.2, proof of Lemma 5.4.2).

Remark 5.2.4. All these assumptions hold for the Oldroyd "8 constants" model, models of Larson, Giesekus, Phan-Thien and Tanner, the Jeffreys model with Oldroyd's derivative (the model from Section 5.1) as well as with Jaumann's or Spriggs' derivative, models of Prandtl and Eyring and any combinations of all these models (see Sections 1.4.4, 1.5.1, 1.5.2).

Theorem 5.2.1 (see [73]). *Let the described assumptions hold true. Let $T > 0$ be an* *arbitrary fixed moment of time. Then, for any*

$$\begin{aligned}
&a \in H^3_V,\\
&\tau_0^k \in H^3_M, \quad k = 1, \dots, r,\\
&f_0 \in L_1(0, T; H^3(\mathbb{R}^n, \mathbb{R}^n)) \cap L_2(0, T; H^2(\mathbb{R}^n, \mathbb{R}^n)),
\end{aligned}$$

there exists a constant $K_5 > 0$, independent on T, such that provided

$$\|a\|_3 + \sum_{k=1}^{r} \|\tau_0^k\|_3 + \|f_0\|_{L_1(0,T;H^3)} < K_5, \tag{5.2.16}$$

problem (5.2.1) – (5.2.10) has a solution in the class

$$u \in L_2(0, T; H^4_V) \cap C([0, T]; H^3_V) \cap W^1_2(0, T; H^2_V), \tag{5.2.17}$$

$$\mathrm{T} \in L_2(0, T; H^3_{M,\mathrm{loc}}), \tag{5.2.18}$$

$$p \in L_2(0, T; H^3_{\mathrm{loc}}(\mathbb{R}^n)). \tag{5.2.19}$$

Furthermore, we have the following information about the constituents of the stress tensor:

$$\tau^0 \in L_2(0, T; H^3_M) \cap C([0, T]; H^2_M) \cap W^1_2(0, T; H^1_M), \tag{5.2.20}$$

$$\tau^k \in L_\infty(0, T; H^3_M) \cap C([0, T]; H^2_M) \cap C^1([0, T]; H^1_M),\ k = 1, \dots, r. \tag{5.2.21}$$

This solution is unique in class (5.2.17) – (5.2.21).

Corollary 5.2.1. *If $f_0 \in L_1(0, +\infty; H^3(\mathbb{R}^n, \mathbb{R}^n)) \cap L_2(0, +\infty; H^2(\mathbb{R}^n, \mathbb{R}^n))$, and*

$$\|a\|_3 + \sum_{k=1}^{r} \|\tau_0^k\|_3 + \|f_0\|_{L_1(0,+\infty;H^3)} < K_5, \tag{5.2.22}$$

then problem (5.2.1) – (5.2.10) has a unique solution in class (5.2.17) – (5.2.21) for every $T > 0$.

The following three sections are devoted to the proof of the theorem.

5.3 Operator treatment of the problem

In this section problem (5.2.1) – (5.2.10), describing the motion of a nonlinear viscoelastic medium, will be rewritten as a Cauchy problem in a Banach space.

Below for simplicity we shall consider the case $r = 1$ (concerning $r > 1$, see Section 5.5.3). Let for brevity $\tau = \tau^1,\ \lambda = \lambda_1,\ \eta = \frac{\eta_1}{\lambda_1}$. Let also

$$\Phi(\mathcal{E}) = \Psi(\mathcal{E}) - 2\eta_0\mathcal{E}. \tag{5.3.1}$$

Introduce the function $g : \mathbb{R}^{n\times n}_S \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}_S$ by the formula

$$g(\xi_1, \xi_2) = \xi_1\Big(\frac{\xi_2 - \xi_2^\top}{2}\Big) - \Big(\frac{\xi_2 - \xi_2^\top}{2}\Big)\xi_1 + \frac{\beta_1(\xi_1, \frac{\xi_2+\xi_2^\top}{2})}{\lambda}.$$

Then

$$g(\tau, \nabla u) = \tau W - W\tau + \frac{\beta_1(\tau, \mathcal{E}(u))}{\lambda}. \tag{5.3.2}$$

Note that $\operatorname{Div}\ pI = \operatorname{grad} p$ and, due to condition (5.2.2), $2\operatorname{Div}\mathcal{E}(u) = \Delta u$. We have from (5.2.1) – (5.2.5), (5.3.1) – (5.3.2):

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} u_i\frac{\partial u}{\partial x_i} + \operatorname{grad} p - \eta_0\Delta u - \operatorname{Div}(\Phi(\mathcal{E}) + \tau) = f_0 \tag{5.3.3}$$

$$\frac{\tau}{\lambda} + \frac{\partial \tau}{\partial t} + \sum_{i=1}^{n} u_i\frac{\partial \tau}{\partial x_i} + g(\tau, \nabla u) = 2\eta\mathcal{E}. \tag{5.3.4}$$

Let $\Phi = (\Phi_{ij})$. Let us introduce the following notations

$$(\widetilde{A}(u)v)_j = \sum_{i,k,l=1}^{n} \frac{\partial \Phi_{ij}}{\partial \xi_{kl}}(\mathcal{E}(u))\frac{\partial \mathcal{E}_{kl}(v)}{\partial x_i}, \quad j = 1, \dots, n, \tag{5.3.5}$$

$$A(u)v = -P\widetilde{A}(u)v + v - \eta_0\Delta v, \tag{5.3.6}$$

$$F_1(u, v) = -P\sum_{i=1}^{n} u_i\frac{\partial v}{\partial x_i}, \tag{5.3.7}$$

$$F(u, \tau) = -\sum_{i=1}^{n} u_i\frac{\partial \tau}{\partial x_i}, \tag{5.3.8}$$

$$\widetilde{F}(u) = F_1(u, u) + u, \tag{5.3.9}$$

$$G(u, \tau) = -g(\tau, \nabla u), \tag{5.3.10}$$

$$N_1(\tau) = P(\operatorname{Div}\ \tau), \tag{5.3.11}$$

$$N_2(u) = 2\eta\mathcal{E}(u), \tag{5.3.12}$$

$$B_0 = I - \Delta, \tag{5.3.13}$$

$$A_0 = A(a), \tag{5.3.14}$$

$$f = Pf_0. \tag{5.3.15}$$

Consider the problem

$$\frac{du}{dt} + A(u)u = \widetilde{F}(u) + N_1(\tau) + f, \tag{5.3.16}$$

$$\frac{d\tau}{dt} + \frac{\tau}{\lambda} = F(u,\tau) + N_2(u) + G(u,\tau), \tag{5.3.17}$$

$$u(0) = a, \qquad \tau(0) = \tau_0. \tag{5.3.18}$$

Remark 5.3.1. Here and below $\frac{d}{dt}$ is not a substantial derivative but the time derivative of a time-dependent function u with values in a Banach space.

Remark 5.3.2. Equation (5.3.16) is a formal consequence of (5.3.3) after application of the Leray projection. Equation (5.3.17) is formally equivalent to (5.3.4).

The statement of Theorem 5.2.1 will be deduced from the following result.

Theorem 5.3.1. *Given $a \in H_V^3$, $\tau_0 \in H_M^3$, $f \in L_1(0,T;H_V^3) \cap L_2(0,T;H_V^2)$, there exists a constant $K_6 > 0$, independent on T, such that provided*

$$\|a\|_3 + \|\tau_0\|_3 + \|f\|_{L_1(0,T;H_V^3)} < K_6 \tag{5.3.19}$$

problem (5.3.16) – (5.3.18) has a unique solution in the class

$$u \in L_2(0,T;H_V^4) \cap C([0,T];H_V^3) \cap W_2^1(0,T;H_V^2), \tag{5.3.20}$$

$$\tau \in L_\infty(0,T;H_M^3) \cap C([0,T];H_M^2) \cap C^1([0,T];H_M^1). \tag{5.3.21}$$

5.4 Auxiliary problem

5.4.1 Solvability of the auxiliary problem

Before proving Theorems 5.2.1 and 5.3.1, we investigate the solvability of an auxiliary problem.

Introduce a family of operators

$$A_\varepsilon(\tau) = \frac{\tau}{\lambda} - \varepsilon\Delta\tau, \ \varepsilon > 0, \tag{5.4.1}$$

and consider the following equation

$$\frac{d\tau}{dt} + A_\varepsilon\tau = F(u,\tau) + N_2(u) + G(u,\tau). \tag{5.4.2}$$

Theorem 5.4.1. *Given* $a \in H_V^4,\ \tau_0 \in H_M^4,\ f \in L_1(0,T;H_V^3) \cap C^1([0,T];H_V^2)$, *there exists a constant* $K_7 > 0$*, independent on* T *and* ε*, such that provided*

$$\|a\|_3 + \|\tau_0\|_3 + \|f\|_{L_1(0,T;H_V^3)} < K_7$$

there is a unique solution of problem (5.3.16), (5.4.2), (5.3.18) in the class

$$\begin{aligned} &u \in C^1([0,T];H_V^2) \cap C([0,T];H_V^4),\\ &\tau \in C^1([0,T];H_M^2) \cap C([0,T];H_M^4). \end{aligned} \tag{5.4.3}$$

In order to get this theorem we have to prove some auxiliary facts.

5.4.2 Operator estimates

We need some estimates for the operators introduced in Section 5.3.

Lemma 5.4.1. *The following estimates take place.*
For $l = 1,2,\ u \in H_V^2,\ v \in H_V^{l+1},\ \tau \in H_M^{l+1}$

$$\|F_1(u,v)\|_l \le K_8 \|u\|_2 \|v\|_{l+1}, \tag{5.4.4}$$

$$\|F(u,\tau)\|_l \le K_8 \|u\|_2 \|\tau\|_{l+1}. \tag{5.4.5}$$

For $l = 2,3,\ u \in H_V^3,\ v \in H_V^{l+1},\ \tau \in H_M^{l+1}$

$$\begin{aligned} |(F_1(u,v),v)_l| &\le K_9 \|\nabla u\|_2 \|\nabla v\|_{l-1}^2,\\ |(F(u,\tau),\tau)_l| &\le K_9 \|\nabla u\|_2 \|\nabla \tau\|_{l-1}^2. \end{aligned} \tag{5.4.6}$$

For $l = 0,1$ *and* $\gamma = \frac{7}{4}$ *or* $l = \gamma = 2$

$$\begin{aligned} \|G(u_1,\tau_1) - G(u_2,\tau_2)\|_l \le K_{10} \cdot (&\|\nabla u_1 - \nabla u_2\|_l + \|\tau_1 - \tau_2\|_l)\\ &\cdot (\|\nabla u_1\|_\gamma + \|\nabla u_2\|_\gamma + \|\tau_1\|_\gamma + \|\tau_2\|_\gamma) \end{aligned} \tag{5.4.7}$$

where $u_1,\ u_2 \in H_V^{\gamma+1},\ \tau_1, \tau_2 \in H_M^\gamma$.
For $u \in H_V^4,\ \tau \in H_M^3$

$$\|G(u,\tau)\|_3 \le K'_{10} \cdot (\|u\|_3 + \|\tau\|_2)(\|\nabla u\|_3 + \|\tau\|_3). \tag{5.4.8}$$

Here K_{10} *and* K'_{10} *depend continuously on* $(\|\nabla u_1\|_\gamma + \|\nabla u_2\|_\gamma + \|\tau_1\|_\gamma + \|\tau_2\|_\gamma)$ *and on* $(\|u\|_3 + \|\tau\|_2)$*, respectively.*

Lemma 5.4.2. *Let* $l = 0,1,2$ *and* $\alpha_0(0) = \frac{11}{8},\ \alpha_0(1) = \frac{7}{8},\ \alpha_0(2) = \frac{1}{2}$.

1) *If*
$$\|B_0^{\alpha_0(l)}v\|_l,\ \|B_0^{\alpha_0(l)}w\|_l < 1$$
then the following estimates take place
$$\begin{aligned}\|(A(v)-A(w))h\|_l &\le K_{11}\|\nabla B_0^{\alpha_0(l)}(v-w)\|_{l-1}\|\nabla h\|_{l+1},\\ \|(A(v)-A(w))h\|_l &\le K_{11}\|B_0^{\alpha_0(l)}(v-w)\|_l\|B_0 h\|_l.\end{aligned} \tag{5.4.9}$$
Furthermore, for $l=0$ there is another variant of the second estimate:
$$\|(A(v)-A(w))h\|_0 \le K_{11}\|v-w\|_{\frac{3}{2}}\|h\|_3. \tag{5.4.10}$$
2) *There exist $K_{12}, K_{13}, K_{14}, K_{15} > 0$ such that provided*
$$\|B_0^{\alpha_0(l)}a\|_l < K_{12}$$
one has
$$(A_0 v, B_0 v)_l \ge K_{13}\|B_0 v\|_l^2, \tag{5.4.11}$$
and provided
$$\|u\|_3,\ \|v\|_3 < K_{14}$$
one has
$$(A(u)u-A(v)v, B_0(u-v))_1 \ge \frac{1}{2}K_{13}\|B_0(u-v)\|_1^2, \tag{5.4.12}$$
$$((A(u)-I)u, B_0 u)_2 \ge K_{15}\|\nabla u\|_3^2. \tag{5.4.13}$$

Proof of Lemma 5.4.1. Using inequality (2.1.21), we have:
$$\|F_1(u,v)\|_1 \le \sum_{i=1}^n \Big\|u_i\frac{\partial v}{\partial x_i}\Big\|_1 \le C\sum_{i=1}^n \|u_i\|_2\Big\|\frac{\partial v}{\partial x_i}\Big\|_1 \le C\|u\|_2\|v\|_2.$$
Using (2.1.22), we have:
$$\|F_1(u,v)\|_2 \le \sum_{i=1}^n \Big\|u_i\frac{\partial v}{\partial x_i}\Big\|_2 \le C\sum_{i=1}^n \|u_i\|_2\Big\|\frac{\partial v}{\partial x_i}\Big\|_2 \le C\|u\|_2\|v\|_3.$$
Estimate (5.4.4) is proved. Similarly one proves (5.4.5).

Let us prove now estimate (5.4.6). We have, using the Leibnitz rule:
$$\begin{aligned}|(F_1(u,v),v)_2| &\le \sum_{|\alpha|\le 2}\sum_{i=1}^n \Big|\Big(D^\alpha\Big(u_i\frac{\partial v}{\partial x_i}\Big), D^\alpha v\Big)_0\Big|\\ &\le \sum_{|\alpha|\le 2}\sum_{i=1}^n \Big|\Big(u_i D^\alpha\frac{\partial v}{\partial x_i}, D^\alpha v\Big)_0\Big| + \sum_{0<|\alpha|\le 2}\sum_{i=1}^n \Big|\Big(D^\alpha u_i\frac{\partial v}{\partial x_i}, D^\alpha v\Big)_0\Big|\\ &\quad + \sum_{|\alpha|=2,|\beta|=1,\beta\le\alpha}\sum_{i=1}^n K_{16}(\alpha,\beta)\Big|\Big(D^\beta u_i D^{\alpha-\beta}\frac{\partial v}{\partial x_i}, D^\alpha v\Big)_0\Big|.\end{aligned}$$

Since $\operatorname{div} u = 0$, integration by parts easily shows that the first sum is equal to zero. With the help of (2.1.20) and Cauchy–Buniakowski inequality we observe that the second sum does not exceed the expression

$$C\sum_{i=1}^{n}\|\nabla u_i\|_2\|\frac{\partial v}{\partial x_i}\|_1\|\nabla v\|_1. \tag{5.4.14}$$

Using inequality (2.1.19), we conclude that the third sum does not exceed an expression of form (5.4.14). Estimate (5.4.6) for the case $l = 2$ is proved. The case $l = 3$ is examined analogously.

Let us prove now estimate (5.4.7).

The conditions on coefficients α_j^k from Section 5.2.3 imply that the function

$$\beta_1 : \mathbb{R}_S^{n\times n}\times\mathbb{R}_S^{n\times n}\to\mathbb{R}_S^{n\times n}$$

is C^3-smooth, and

$$\beta_1(0,0) = 0, \quad \nabla_\xi\,\beta_1(0,0) = 0.$$

Then the function

$$g : \mathbb{R}_S^{n\times n}\times\mathbb{R}^{n\times n}\to\mathbb{R}_S^{n\times n}$$

introduced in Section 5.3 is also C^3 - smooth, and

$$g(0,0) = 0, \quad \nabla_\xi\, g(0,0) = 0.$$

We have:

$$\|G(u_1,\tau_1) - G(u_2,\tau_2)\|_l \le \sum_{|\alpha|\le l}\|D^\alpha(G(u_1,\tau_1) - G(u_2,\tau_2))\|_0.$$

First we estimate the term with $|\alpha| = 0$. Taking into account embedding (2.1.18) and the Lagrange theorem we have:

$$\begin{aligned}
&\|G(u_1,\tau_1) - G(u_2,\tau_2)\|_0 = \|g(\nabla u_1,\tau_1) - g(\nabla u_2,\tau_2)\|_0\\
&\le \sup_{\substack{\|\xi^1\|\le\|\nabla u_1\|_{L_\infty}+\|\nabla u_2\|_{L_\infty}\\ \|\xi^2\|\le\|\tau_1\|_{L_\infty}+\|\tau_2\|_{L_\infty}}} |\nabla_\xi g(\xi^1,\xi^2)|(\|\nabla(u_1-u_2)\|_0 + \|\tau_1-\tau_2\|_0)\\
&= \sup_{\substack{\|\xi^1\|\le\|\nabla u_1\|_{L_\infty}+\|\nabla u_2\|_{L_\infty}\\ \|\xi^2\|\le\|\tau_1\|_{L_\infty}+\|\tau_2\|_{L_\infty}}} |\nabla_\xi g(\xi^1,\xi^2) - \nabla_\xi g(0,0)|(\|\nabla(u_1-u_2)\|_0 + \|\tau_1-\tau_2\|_0)\\
&\le C(\|\nabla u_1\|_{L_\infty} + \|\nabla u_2\|_{L_\infty} + \|\tau_1\|_{L_\infty} + \|\tau_2\|_{L_\infty})(\|\nabla(u_1-u_2)\|_0 + \|\tau_1-\tau_2\|_0)\\
&\le C\big(\|\nabla u_1\|_{\frac{7}{4}} + \|\nabla u_2\|_{\frac{7}{4}} + \|\tau_1\|_{\frac{7}{4}} + \|\tau_2\|_{\frac{7}{4}}\big)(\|\nabla(u_1-u_2)\|_0 + \|\tau_1-\tau_2\|_0).
\end{aligned}$$

Note that for $u\in H_V^2$, $\tau\in H_V^1$ one has

$$\nabla g(\nabla u,\tau) = \frac{\partial g}{\partial\xi_{\nabla u}}(\nabla u,\tau)\nabla\nabla u + \frac{\partial g}{\partial\xi_\tau}(\nabla u,\tau)\nabla\tau,$$

and let us denote it by
$$\nabla_\xi g(\nabla u, \tau)(\nabla^2 u, \nabla \tau).$$
Using Hölder's inequality (2.1.1) in the form
$$\|\psi_1 \psi_2\|_0 \le \|\psi_1\|_{L_6} \|\psi_2\|_{L_3}$$
and the embeddings
$$H^{1/2} \subset L_3, \quad H^1 \subset L_6, \quad H^{7/4} \subset L_\infty,$$
we estimate the first derivatives:
$$\begin{aligned}
&\|\nabla(G(u_1,\tau_1) - G(u_2,\tau_2))\|_0 \\
&\quad = \|\nabla_\xi g(\nabla u_1,\tau_1)(\nabla^2 u_1, \nabla\tau_1) - \nabla_\xi g(\nabla u_2,\tau_2)(\nabla^2 u_2, \nabla\tau_2)\|_0 \\
&\quad \le \|\nabla_\xi g(\nabla u_1,\tau_1)(\nabla^2 (u_1 - u_2), \nabla(\tau_1 - \tau_2))\|_0 \\
&\qquad + \|(\nabla_\xi g(\nabla u_1,\tau_1) - \nabla_\xi g(\nabla u_2,\tau_2))(\nabla^2 u_2, \nabla\tau_2)\|_0 \\
&\quad \le \|\nabla_\xi g(\nabla u_1,\tau_1)\|_{L_\infty}(\|\nabla(u_1 - u_2)\|_1 + \|\tau_1 - \tau_2\|_1) \\
&\qquad + \|\nabla_\xi g(\nabla u_1,\tau_1) - \nabla_\xi g(\nabla u_2,\tau_2)\|_{L_6}(\|\nabla^2 u_2\|_{L_3} + \|\nabla\tau_2\|_{L_3}) \\
&\quad \le \|\nabla_\xi g(\nabla u_1,\tau_1) - \nabla_\xi g(0,0)\|_{L_\infty}(\|\nabla(u_1 - u_2)\|_1 + \|\tau_1 - \tau_2\|_1) \\
&\qquad + \|\nabla_\xi g(\nabla u_1,\tau_1) - \nabla_\xi g(\nabla u_2,\tau_2)\|_{L_6}(\|\nabla^2 u_2\|_{L_3} + \|\nabla\tau_2\|_{L_3}) \\
&\quad \le C(\|\nabla u_1\|_{L_\infty} + \|\tau_1\|_{L_\infty})(\|\nabla(u_1 - u_2)\|_1 + \|\tau_1 - \tau_2\|_1) \\
&\qquad + C(\|\nabla(u_1 - u_2)\|_{L_6} + \|\tau_1 - \tau_2\|_{L_6})(\|\nabla^2 u_2\|_{\frac12} + \|\nabla\tau_2\|_{\frac12}) \\
&\quad \le C\big(\|\nabla(u_1 - u_2)\|_1 + \|\tau_1 - \tau_2\|_1\big)\big(\|\nabla u_1\|_{\frac74} + \|\tau_1\|_{\frac74} + \|\nabla u_2\|_{\frac74} + \|\tau_2\|_{\frac74}\big).
\end{aligned}$$

In the same way the terms $\|D^\alpha(G(u_1,\tau_1) - G(u_2,\tau_2))\|_0$ with $|\alpha| = 2$ are estimated and we obtain estimate (5.4.7).

Let us turn to estimate (5.4.8). We have:
$$\begin{aligned}
\|G(u,\tau)\|_3 &\le \|g(\nabla u,\tau)\|_2 + \|\nabla^3 g(\nabla u,\tau)\|_0 \\
&\le \|g(\nabla u,\tau)\|_2 + \|\nabla_\xi g(\nabla u,\tau)(\nabla^4 u, \nabla^3\tau)\|_0 \\
&\quad + \|\nabla_\xi^2 g(\nabla u,\tau)(\nabla^3 u, \nabla^2\tau)(\nabla^2 u, \nabla\tau)\|_0 \\
&\quad + \|\nabla_\xi^2 g(\nabla u,\tau)(\nabla^2 u, \nabla\tau)(\nabla^3 u, \nabla^2\tau)\|_0 \\
&\quad + \|\nabla_\xi^3 g(\nabla u,\tau)(\nabla^2 u, \nabla\tau)(\nabla^2 u, \nabla\tau)(\nabla^2 u, \nabla\tau)\|_0.
\end{aligned}$$

All these items except the last one are bounded at least by
$$C \cdot (\|u\|_3 + \|\tau\|_2)(\|\nabla u\|_3 + \|\tau\|_3)$$

where C depends on $\|u\|_3 + \|\tau\|_2$ (it may be checked in a similar way as above). Using Hölder's inequality in the form

$$\|\psi_1\psi_2\psi_3\|_0 \le \|\psi_1\|_{L_6}\|\psi_2\|_{L_6}\|\psi_3\|_{L_6},$$

we conclude that the last item does not exceed

$$\begin{aligned}\sup_{\substack{\|\xi^1\|\le\|\nabla u\|_{L_\infty}\\ \|\xi^2\|\le\|\tau\|_{L_\infty}}} &|\nabla^3_\xi g(\xi^1,\xi^2)|(\|\nabla^2 u\|_{L_6} + \|\nabla\tau\|_{L_6})^3\\ &\le C\cdot(\|\nabla^2 u\|_1 + \|\nabla\tau\|_1)^3\\ &\le C\cdot(\|u\|_3 + \|\tau\|_2)(\|\nabla u\|_3 + \|\tau\|_3)\end{aligned}$$

where both C depend on $\|u\|_3 + \|\tau\|_2$. □

Proof of Lemma 5.4.2. The C^4-smoothness of coefficients φ_i (see Section 5.2.3) implies that the function

$$\Phi : \mathbb{R}^{n\times n}_S \to \mathbb{R}^{n\times n}_S$$

introduced by formula (5.3.1) is C^4-smooth. Moreover, it is easy to check that

$$\Phi(0) = 0, \nabla_\xi\Phi(0) = 0.$$

We have:

$$\|(A(v) - A(w))h\|_l = \|(\widetilde{A}(v) - \widetilde{A}(w))h\|_l \le \sum_{|\alpha|\le l}\|D^\alpha(\widetilde{A}(v) - \widetilde{A}(w))h\|_0.$$

First we estimate the term with $|\alpha| = 0$:

$$\begin{aligned}&\|(\widetilde{A}(v) - \widetilde{A}(w))h\|_0\\ &\quad\le \sum_{i,j,k,l=1}^n \|\left(\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(v)) - \frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(w))\right)\frac{\partial\mathcal{E}_{kl}(h)}{\partial x_i}\|_0\\ &\quad\le \sum_{i,j,k,l=1}^n \left\|\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(v)) - \frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(w))\right\|_{L_\infty}\|\frac{\partial\mathcal{E}_{kl}(h)}{\partial x_i}\|_0\\ &\quad\le C\|\mathcal{E}(v) - \mathcal{E}(w)\|_{L_\infty}\|\nabla h\|_1\\ &\quad\le C\|\mathcal{E}(v) - \mathcal{E}(w)\|_{\frac{7}{4}}\|\nabla h\|_1 \le C\|\nabla B_0^{\frac{11}{8}}(v-w)\|_{-1}\|\nabla h\|_1.\end{aligned} \tag{5.4.15}$$

Furthermore,

$$
\begin{aligned}
&\|(\widetilde{A}(v)-\widetilde{A}(w))h\|_0 \\
&\quad\le \sum_{i,j,k,l=1}^{n} \Big\| \left(\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(v))-\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(w))\right)\frac{\partial\mathcal{E}_{kl}(h)}{\partial x_i}\Big\|_0 \\
&\quad\le \sum_{i,j,k,l=1}^{n} \left\|\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(v))-\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(w))\right\|_{L_3}\Big\|\frac{\partial\mathcal{E}_{kl}(h)}{\partial x_i}\Big\|_{L_6} \qquad (5.4.16)\\
&\quad\le C\|\mathcal{E}(v)-\mathcal{E}(w)\|_{L_3}\Big\|\frac{\partial\mathcal{E}_{kl}(h)}{\partial x_i}\Big\|_1 \\
&\quad\le C\|v-w\|_{\frac{3}{2}}\|h\|_3,
\end{aligned}
$$

and this implies (5.4.10).

For $|\alpha|=1$ we obtain

$$
\begin{aligned}
&\|D^\alpha((\widetilde{A}(v)-\widetilde{A}(w))h)\|_0 \\
&\le \sum_{i,j,k,l=1}^{n}\Bigg(\Big\|D^\alpha\left(\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(v))-\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(w))\right)\frac{\partial\mathcal{E}_{kl}(h)}{\partial x_i}\Big\|_0 \\
&\qquad +\Big\|\left(\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(v))-\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(w))\right)D^\alpha\frac{\partial\mathcal{E}_{kl}(h)}{\partial x_i}\Big\|_0\Bigg) \\
&\le \sum_{i,j,k,l=1}^{n}\Bigg(\Big\|D^\alpha\left(\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(v))-\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(w))\right)\Big\|_{L_3}\Big\|\frac{\partial\mathcal{E}_{kl}(h)}{\partial x_i}\Big\|_{L_6} \\
&\qquad +\Big\|\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(v))-\frac{\partial\Phi_{ij}}{\partial\xi_{kl}}(\mathcal{E}(w))\Big\|_{L_\infty}\Big\|D^\alpha\frac{\partial\mathcal{E}_{kl}(h)}{\partial x_i}\Big\|_0\Bigg) \\
&\le \sum_{i,j,k,l=1}^{n} C\Bigg(\Big\|\sum_{k_1,l_1=1}^{n}\frac{\partial^2\Phi_{ij}}{\partial\xi_{kl}\partial\xi_{k_1l_1}}(\mathcal{E}(v))D^\alpha\mathcal{E}_{k_1l_1}(v) \\
&\qquad -\frac{\partial^2\Phi_{ij}}{\partial\xi_{kl}\partial\xi_{k_1l_1}}(\mathcal{E}(w))D^\alpha\mathcal{E}_{k_1l_1}(w)\Big\|_{L_3}\|\nabla h\|_2+\|\mathcal{E}(v)-\mathcal{E}(w)\|_{L_\infty}\|\nabla h\|_2\Bigg) \\
&\le \sum_{i,j,k,l=1}^{n} C\Bigg(\sum_{k_1,l_1=1}^{n}\Bigg[\Big\|\frac{\partial^2\Phi_{ij}}{\partial\xi_{kl}\partial\xi_{k_1l_1}}(\mathcal{E}(v))\Big\|_{L_\infty}\|D^\alpha(\mathcal{E}_{k_1l_1}(v)-\mathcal{E}_{k_1l_1}(w))\|_{L_3} \\
&\qquad +\Big\|\frac{\partial^2\Phi_{ij}}{\partial\xi_{kl}\partial\xi_{k_1l_1}}(\mathcal{E}(v))-\frac{\partial^2\Phi_{ij}}{\partial\xi_{kl}\partial\xi_{k_1l_1}}(\mathcal{E}(w))\Big\|_{L_\infty}\|D^\alpha\mathcal{E}_{k_1l_1}(w)\|_{L_3}\Bigg] \\
&\qquad +\|\mathcal{E}(v)-\mathcal{E}(w)\|_{L_\infty}\Bigg)\|\nabla h\|_2
\end{aligned}
$$

$$
\begin{aligned}
&\le C\Big[\Big(\Big\|\frac{\partial^2\Phi_{ij}}{\partial\xi_{kl}\partial\xi_{k_1l_1}}(\mathcal{E}(v)) - \frac{\partial^2\Phi_{ij}}{\partial\xi_{kl}\partial\xi_{k_1l_1}}(0)\Big\|_{L_\infty} \\
&\quad + \Big\|\frac{\partial^2\Phi_{ij}}{\partial\xi_{kl}\partial\xi_{k_1l_1}}(0)\Big\|_{L_\infty}\Big)\|\nabla(v-w)\|_{\frac{7}{4}} + \|\mathcal{E}(v)-\mathcal{E}(w)\|_{L_\infty}\|\nabla w\|_{\frac{7}{4}} \\
&\quad + \|\mathcal{E}(v)-\mathcal{E}(w)\|_{L_\infty}\Big]\|\nabla h\|_2 \\
&\le C\Big[(\|\mathcal{E}(v)\|_{L_\infty} + C)\|\nabla(v-w)\|_{\frac{7}{4}} + \|\mathcal{E}(v)-\mathcal{E}(w)\|_{L_\infty}\|w\|_{\frac{11}{4}} \\
&\quad + \|\mathcal{E}(v)-\mathcal{E}(w)\|_{\frac{7}{4}}\Big]\|\nabla h\|_2 \\
&\le C\Big(\|v\|_{\frac{11}{4}}\|\nabla(v-w)\|_{\frac{7}{4}} + \|\nabla(v-w)\|_{\frac{7}{4}} + \|\nabla(v-w)\|_{\frac{7}{4}}\|w\|_{\frac{11}{4}}\Big)\|\nabla h\|_2 \\
&\le C\|\nabla B_0^{\frac{7}{8}}(v-w)\|_0\|\nabla h\|_2.
\end{aligned}
\tag{5.4.17}
$$

For $|\alpha| = 2$ in a similar way one obtains

$$
\big\|D^\alpha\big((\widetilde{A}(v) - \widetilde{A}(w))h\big)\big\|_0 \le C\|\nabla B_0^{1/2}(v-w)\|_1\|\nabla h\|_3. \tag{5.4.18}
$$

Estimates (5.4.15), (5.4.17), (5.4.18) yield estimate (5.4.9) for all l.

Notations (5.3.5), (5.3.6) imply $A(0) = I - \eta_0\Delta$. Therefore

$$
(A(0)v, B_0v)_l \ge 2K_{13}\|B_0v\|_l^2
$$

for some $K_{13} > 0$.

On the other hand, for K_{12} small enough, estimate (5.4.9) yields

$$
\begin{aligned}
\|((A_0 - A(0))v, B_0v)_l\| &\le \|(A(a) - A(0))v\|_l\|B_0v\|_l \\
&\le K_{11}\|B_0^{\alpha_0(l)}a\|_l\|B_0v\|_l^2 \le K_{13}\|B_0v\|_l^2
\end{aligned}
$$

what implies (5.4.11).

Let $\|u\|_3,\ \|v\|_3 < K_{14}$ and K_{14} be small enough. Then estimate (5.4.11) yields

$$
(A(u)(u-v), B_0(u-v))_1 \ge K_{13}\|B_0(u-v)\|_1^2.
$$

But (5.4.9) implies

$$
\begin{aligned}
|((A(u) - A(v))v, B_0(u-v))_1| &\le K_{11}\|B_0^{\frac{7}{8}}(u-v)\|_1\|B_0(u-v)\|_1^2 \\
&\le \frac{K_{13}}{2}\|B_0(u-v)\|_1^2
\end{aligned}
$$

for K_{14} small enough, and it yields (5.4.12).

Now, note that

$$
((A(0) - I)u, B_0u)_2 = -\eta_0(\Delta u, B_0u)_2 \ge 2K_{15}\|\nabla u\|_3^2
$$

for some $K_{15} > 0$.

But estimate (5.4.9) and triangle inequality yield

$$
\begin{aligned}
|((A(u) - A(0))u, B_0 u)|_2 &\le K_{11} \|\nabla B_0^{\frac{1}{2}} u\|_1 \|\nabla u\|_3 \|(I - \Delta)u\|_2 \\
&\le C(\|\nabla u\|_2 \|\nabla u\|_3 \|u\|_2 + \|u\|_3 \|\nabla u\|_3 \|\Delta u\|_2) \\
&\le K_{15} \|\nabla u\|_3^2
\end{aligned}
$$

for K_{14} small enough. This implies (5.4.13). □

5.4.3 Properties of the operator A_0

Lemma 5.4.3. *Under the conditions and notations of Lemma 5.4.2 there exist constants K_{17} and K_{18} such that provided*

$$
\|B_0^{\alpha_0(l)} a\|_l < K_{12}, \quad \lambda \in \mathbb{C}, \quad \operatorname{Re} \lambda \ge 0,
$$

the following estimates are valid:

$$
\|(A_0 + \lambda I)v\|_l \le K_{17}(\lambda) \|B_0 v\|_l, \tag{5.4.19}
$$

$$
\|(A_0 + \lambda I)v\|_l \ge K_{18} \|B_0 v\|_l. \tag{5.4.20}
$$

Under the same conditions and $l = 1, 2$ the operator

$$
A_0 : H_V^{l+2} \simeq D(A_0) \subset H_V^l \to H_V^l
$$

is strongly positive, and the constant K_1 in (3.1.11) does not depend on a (but may depend on K_{12}).

Remark 5.4.1. Here we have to use complexifications of Sobolev spaces and differential operators (cf. Remark 3.1.2).

Proof of Lemma 5.4.3. Let $\|B_0^{\alpha_0(l)} a\|_l < K_{12}$. Then estimate (5.4.9) implies

$$
\|(A(a) - A(0))v\|_l \le C \|B_0 v\|_l.
$$

But for all $\lambda \in \mathbb{C}, \operatorname{Re} \lambda \ge 0$:

$$
\|(A(0) + \lambda I)v\|_l = \|(-\eta_0 \Delta + (\lambda + 1)I)v\|_l \le K_{18}(\lambda) \|B_0 v\|_l,
$$

and these two estimates yield (5.4.19).

From (5.4.11) we have:

$$
|((A_0 + \lambda I)v, B_0 v)_l| \ge \operatorname{Re}((A_0 + \lambda I)v, B_0 v)_l \ge K_{13} \|B_0 v\|_l^2. \tag{5.4.21}
$$

This estimate and the Cauchy–Buniakowski inequality imply (5.4.20).

From (5.4.20) it follows that $\mathrm{Ker}(A_0+\lambda I) = \{0\}$. Let us show that $\mathrm{Im}(A_0+\lambda I)$ is dense in H_V^l. If not, then in H_V^l there is a nonzero vector h orthogonal to $\mathrm{Im}(A_0+\lambda I)$. Let $\zeta = B_0^{-1}h \in D(A_0)$. Then (5.4.21) implies

$$\mathrm{Re}((A_0 + \lambda I)\zeta, B_0\zeta)_l \geq K_{13}\|B_0\zeta\|_l^2 > 0. \tag{5.4.22}$$

But since $(A_0 + \lambda I)\zeta \in \mathrm{Im}(A_0 + \lambda I)$ and $B_0\zeta = h$, the left-hand side of (5.4.22) is equal to zero. Hence, $\mathrm{Im}(A_0 + \lambda I)$ is dense in H_V^l.

Let us show now that $\mathrm{Im}(A_0 + \lambda I)$ is closed. Let $h_i \underset{i\to\infty}{\to} h_0,\ h_i \in \mathrm{Im}(A_0 + \lambda I)$. We have to show that $h_0 \in \mathrm{Im}(A_0 + \lambda I)$. Note that there exist ζ_i such that $(A_0 + \lambda I)\zeta_i = h_i$. Then the sequence $(A_0 + \lambda I)\zeta_i$ converges in H_V^l, and from (5.4.20) it follows that $B_0\zeta_i$ also converges in H_V^l: $B_0\zeta_i \to \xi_0$. Let $\zeta_0 = B_0^{-1}\xi_0 \in D(A_0)$. We have: $B_0(\zeta_i - \zeta_0) \to 0$. And estimate (5.4.19) yields $(A_0 + \lambda I)(\zeta_i - \zeta_0) \to 0$, i.e. $h_0 = (A_0 + \lambda I)\zeta_0$.

Thus, $A_0 + \lambda I$ is surjective.

Now, for $|\lambda| \geq 1, \mathrm{Re}\,\lambda \geq 0,\ u \in D(A_0)$ we obtain

$$\begin{aligned}
(u,u)_l &= \mathrm{Re}\,\frac{1}{|\lambda|^2}(\lambda u, \lambda u)_l \\
&\leq \mathrm{Re}\,\frac{1}{|\lambda|^2}(\lambda u, \lambda u)_l + \mathrm{Re}\,\frac{1}{|\lambda|^2}(A_0 u, \lambda B_0 u)_{l-1} \\
&\quad + \mathrm{Re}\,\frac{1}{|\lambda|^2}(\lambda B_0 u, A_0 u)_{l-1} + \mathrm{Re}\,\frac{1}{|\lambda|^2}(A_0 u, A_0 u)_l \\
&= \mathrm{Re}\,\frac{1}{|\lambda|^2}((A_0 + \lambda I)u, (A_0 + \lambda I)u)_l
\end{aligned}$$

whence

$$\|u\|_l \leq \frac{1}{|\lambda|}\|(A_0 + \lambda I)u\|_l \leq \frac{2}{1 + |\lambda|}\|(A_0 + \lambda I)u\|_l. \tag{5.4.23}$$

Similarly, for $|\lambda| \leq 1, \mathrm{Re}\,\lambda \geq 0$, by (5.4.20)

$$\|u\|_l \leq \|B_0 u\|_l \leq K_{18}^{-1}\|(A_0 + \lambda I)u\|_l \leq \frac{2}{1 + |\lambda|}K_{18}^{-1}\|(A_0 + \lambda I)u\|_l.$$

This estimate and (5.4.23) imply

$$\|(A_0 + \lambda I)^{-1}\| \leq \frac{C}{1 + |\lambda|}, \quad \mathrm{Re}\,\lambda \geq 0,$$

with C independent of a. □

5.4.4 Uniqueness lemma

Lemma 5.4.4. *There exists a constant $K_{19} > 0$, independent of T and ε, such that if a solution (u_1, τ_1) of problem (5.3.16), (5.4.2), (5.3.18) exists in the class*

$$\begin{aligned} u_1 &\in C^1([0,T]; H^1_V) \cap C([0,T]; H^3_V), \\ \tau_1 &\in C^1([0,T]; H^1_M) \cap C([0,T]; H^3_M), \end{aligned} \tag{5.4.24}$$

and

$$\|u_1(t)\|_3 < K_{19}, \quad t \in [0,T], \tag{5.4.25}$$

then it is unique in class (5.4.24).

Proof. Assume that in addition to (u_1, τ_1) there is another solution (u_2, τ_2) of problem (5.3.16), (5.4.2), (5.3.18).

Suppose first that

$$\|u_2(t)\|_3 \le K_{19}, \quad t \in [0,T] \tag{5.4.26}$$

(K_{19} will be defined below).

The following identity takes place:

$$(N_1(\tau), u)_m + \frac{1}{2\eta}(N_2(u), \tau)_m = 0, \quad u \in H^{m+1}_V, \quad \tau \in H^{m+1}_M, \quad m = 0, 1, \ldots \tag{5.4.27}$$

Indeed,

$$\begin{aligned} &(u, N_1(\tau))_m + \frac{1}{2\eta}(\tau, N_2(u))_m \\ &= \sum_{i,j=1}^{n} \Big[\big(u_i, \frac{\partial \tau_{ij}}{\partial x_j}\big)_m + \big(\tau_{ij}, \frac{1}{2}\big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\big)\big)_m\Big] \\ &= \sum_{i,j=1}^{n} \Big[\big(u_i, \frac{\partial \tau_{ij}}{\partial x_j}\big)_m + \big(\tau_{ij}, \frac{\partial u_i}{\partial x_j}\big)_m\Big] \\ &= \sum_{i,j=1, |\alpha| \le m}^{n} \int_{\mathbb{R}^n} \Big(D^\alpha u_i D^\alpha \frac{\partial \tau_{ij}}{\partial x_j} + D^\alpha \tau_{ij} D^\alpha \frac{\partial u_i}{\partial x_j}\Big)\, dx = 0 \end{aligned}$$

by Green's formula (cf. Lemma 6.1.1).

Let $w = u_1 - u_2$, $\sigma_* = \tau_1 - \tau_2$. Substitute (u_1, τ_1) and (u_2, τ_2) into (5.3.16) and take the difference of the obtained equalities. Treat (5.4.2) in the same way. We have:

$$\begin{aligned} \frac{dw(t)}{dt} &= A(u_2)u_2 - A(u_1)u_1 + w + F_1(w, u_1) + F_1(u_2, w) + N_1(\sigma_*) \\ \frac{d\sigma_*(t)}{dt} &= -A_\varepsilon(\sigma_*) + F(w, \tau_1) + F(u_2, \sigma_*) + N_2(w) + G(u_1, \tau_1) - G(u_2, \tau_2). \end{aligned}$$

Taking the scalar product of the first equation with $B_0 w(t)$ in H^1_V and of the second one with $\frac{1}{2\eta} B_0\sigma_*(t)$ in H^1_M at every $t \in [0,T]$, adding the obtained equalities and taking into account (5.4.27) we get

$$\begin{aligned}
&\Big(\frac{dw}{dt}, B_0 w\Big)_1 + \frac{1}{2\eta}\Big(\frac{d\sigma_*}{dt}, B_0\sigma_*\Big)_1 \\
&\quad = -(A(u_1)u_1 - A(u_2)u_2, B_0 w)_1 + (w, B_0 w)_1 + (F_1(w,u_1), w)_2 \\
&\qquad + (F_1(u_2,w), w)_2 + \frac{1}{2\eta}[-(A_\varepsilon(\sigma_*), B_0\sigma_*)_1 + (F(w,\tau_1),\sigma_*)_2 \\
&\qquad + (F(u_2,\sigma_*),\sigma_*)_2 + (G(u_1,\tau_1) - G(u_2,\tau_2),\sigma_*)_2].
\end{aligned}$$

Obviously,

$$(A_\varepsilon(\sigma_*), B_0\sigma_*)_1 = \Big(\frac{\sigma_*}{\lambda} - \varepsilon\Delta\sigma_*, \sigma_*\Big)_2 \geq \frac{1}{\lambda}\|\sigma_*\|_2.$$

Now, using estimates (5.4.4) – (5.4.7), (5.4.12) (we can assume K_{19} to be small enough so that estimate (5.4.12) is also valid) we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w\|_2^2 + \frac{1}{4\eta}\frac{d}{dt}\|\sigma_*\|_2^2 + \frac{1}{2}K_{13}\|w\|_3^2 + \frac{1}{2\eta\lambda}\|\sigma_*\|_2^2 \\
&\quad \leq K_{20}[\|w\|_2^2(1 + \|u_1\|_3 + \|u_2\|_3) + \|w\|_2\|\tau_1\|_3\|\sigma_*\|_2 + \|u_2\|_3\|\sigma_*\|_2^2 \\
&\qquad + \|\sigma_*\|_2(\|\sigma_*\|_2 + \|w\|_3)(\|\tau_1\|_2 + \|\tau_2\|_2 + \|u_1\|_3 + \|u_2\|_3)].
\end{aligned}$$

It is obvious that there exists K_{21} such that

$$\begin{aligned}
&\frac{1}{2}K_{13}\|w\|_3^2 + K_{21}\|\sigma_*\|_2^2 \\
&\qquad - K_{20}\|\sigma_*\|_2\|w\|_3(\|\tau_1\|_2 + \|\tau_2\|_2 + \|u_1\|_3 + \|u_2\|_3) \geq 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\Big(\frac{d}{dt}\|w\|_2^2 + \frac{1}{2\eta}\frac{d}{dt}\|\sigma_*\|_2^2\Big) \\
&\quad \leq \Big(K_{21} - \frac{1}{2\eta\lambda}\Big)\|\sigma_*\|_2^2 + K_{20}[\|w\|_2^2(1 + \|u_1\|_3 + \|u_2\|_3) + \|w\|_2\|\tau_1\|_3\|\sigma_*\|_2 \\
&\qquad + \|\sigma_*\|_2^2\|u_2\|_3 + \|\sigma_*\|_2^2(\|\tau_1\|_2 + \|\tau_2\|_2 + \|u_1\|_3 + \|u_2\|_3)] \\
&\quad \leq C\Big(\|w\|_2^2 + \frac{1}{2\eta}\|\sigma_*\|_2^2\Big).
\end{aligned}$$

Since $w(0) = 0,\ \sigma_*(0) = 0$, by the Gronwall lemma $w \equiv 0,\ \tau \equiv 0$.

If (5.4.26) is not valid for some t, let t_1 be the infimum of such t. It is evident that $\|u_2(t_1)\|_3 = K_{19}$. But on the other hand, we have just proved that on $[0,t_1]$ the solution is unique. Hence $u_1(t_1) = u_2(t_1),\ \tau_1(t_1) = \tau_2(t_1)$ what contradicts (5.4.25). □

5.4.5 A priori estimate

Lemma 5.4.5. *There exists a constant $K_{22} > 0$, independent of T and ε, such that for any positive number $\kappa < K_{22}$, and*

$$\|a\|_3 + \frac{1}{\sqrt{2\eta}}\|\tau_0\|_3 + \|f\|_{L_1(0,T;H^3_V)} < \kappa$$

one has the following bounds for every solution u of (5.3.16), (5.4.2), (5.3.18) in class (5.4.3):

$$\|u(t)\|_3^2 + \frac{1}{2\eta}\|\tau(t)\|_3^2 < \kappa^2 < K_{22}^2, \tag{5.4.28}$$

$$\int_0^T \|\nabla u\|_3^2\,ds < \frac{3\kappa^2}{K_{15}}. \tag{5.4.29}$$

Proof. Denote by $t_1 = t_1(\kappa)$ the infimum of those t at which (5.4.28) is not valid (K_{22} will be defined later, κ is an arbitrary positive number which is less than K_{22}).

Taking the scalar product of (5.3.16) with $B_0u(t)$ in H^2_V and of (5.4.2) with $\frac{1}{2\eta}B_0\tau(t)$ in H^2_M at every $t \in [0, t_1]$, adding the obtained equalities and taking into account (5.4.27) we get

$$\begin{aligned}
&\Big(\frac{du}{dt}, B_0u\Big)_2 + \frac{1}{2\eta}\Big(\frac{d\tau}{dt}, B_0\tau\Big)_2 \\
&\quad = -(A(u)u, B_0u)_2 + (u, B_0u)_2 + (F_1(u,u), B_0u)_2 \\
&\qquad + \frac{1}{2\eta}[-(A_\varepsilon(\tau), B_0\tau)_2 + (F(u,\tau), B_0\tau)_2 + (G(u,\tau), B_0\tau)_2] + (f, B_0u)_2.
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|_3^2 + \frac{1}{4\eta}\frac{d}{dt}\|\tau\|_3^2 &\le -((A(u) - I)u, B_0u)_2 + (F_1(u,u), u)_3 - \frac{1}{2\eta\lambda}\|\tau\|_3^2 \\
&\quad + \frac{1}{2\eta}(F(u,\tau),\tau)_3 + \frac{1}{2\eta}(G(u,\tau),\tau)_3 + (f, u)_3.
\end{aligned}$$

Using (5.4.6), (5.4.8), (5.4.13) (we can assume K_{22} to be small enough so that estimate (5.4.13) is also valid) we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|_3^2 + \frac{1}{4\eta}\frac{d}{dt}\|\tau\|_3^2 \\
&\quad \le -K_{15}\|\nabla u\|_3^2 - \frac{1}{2\eta\lambda}\|\tau\|_3^2 + K_9\|\nabla u\|_2^3 + \frac{K_9}{2\eta}\|\nabla u\|_2\|\tau\|_3^2 \\
&\qquad + \frac{1}{2\eta}K'_{10}\cdot(\|\nabla u\|_3 + \|\tau\|_3)(\|u\|_3 + \|\tau\|_2)\|\tau\|_3 + \|f\|_3\|u\|_3.
\end{aligned}$$

Note that K'_{10} is uniformly bounded for $\|u(t)\|_3 + \|\tau(t)\|_2$ is bounded.

For K_{22} small enough this yields

$$\frac{1}{2}\frac{d}{dt}\|u\|_3^2 + \frac{1}{4\eta}\frac{d}{dt}\|\tau\|_3^2 \le \|f\|_3\|u\|_3 - \frac{1}{2}K_{15}\|\nabla u\|_3^2. \tag{5.4.30}$$

Let $\psi(t) = \sqrt{\|u\|_3^2(t) + \frac{1}{2\eta}\|\tau\|_3^2(t)}$. Then from (5.4.30) it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\psi^2(t) \le \|f\|_3\psi(t), \qquad t \in [0, t_1], \\ &\frac{d}{dt}\psi(t) \le \|f\|_3. \end{aligned}$$

Integrate from 0 to t_1:

$$\begin{aligned} \psi(t_1) &\le \sqrt{\|a\|_3^2 + \frac{1}{2\eta}\|\tau_0\|_3^2} + \int_0^{t_1} \|f\|_3(s)ds \\ &\le \|a\|_3 + \frac{1}{\sqrt{2\eta}}\|\tau_0\|_3 + \int_0^T \|f\|_3(s)ds < \kappa, \end{aligned}$$

which contradicts the definition of t_1. Thus, (5.4.28) is valid for all $t \in [0, T]$.

Integrating (5.4.30) from 0 to T we obtain

$$\frac{1}{2}\|u\|_3^2(T) + \frac{1}{4\eta}\|\tau\|_3^2(T) - \frac{1}{2}\|a\|_3^2 - \frac{1}{4\eta}\|\tau_0\|_3^2 \le \int_0^T (\|f\|_3\|u\|_3 - \frac{1}{2}K_{15}\|\nabla u\|_3^2)\, ds.$$

This inequality and (5.4.28) imply

$$\begin{aligned} \frac{1}{2}K_{15}\int_0^T \|\nabla u\|_3^2\, ds &\le \frac{1}{2}\|a\|_3^2 + \frac{1}{4\eta}\|\tau_0\|_3^2 + \int_0^T \|f\|_3\|u\|_3\, ds \\ &\le \frac{1}{2}\kappa^2 + \kappa\int_0^T \|f\|_3\, ds \le \frac{3}{2}\kappa^2, \end{aligned}$$

and we get estimate (5.4.29). □

5.4.6 Proof of Theorem 5.4.1

Since the operators F_1 and F are bilinear, estimates (5.4.4) and (5.4.5) imply

$$\|\widetilde{F}(u) - \widetilde{F}(u)\|_l \le C\|u - v\|_{l+1}, \tag{5.4.31}$$

$$\|F(u,\tau) - F(v,\tau_*)\|_l \le C(\|u - v\|_{l+1} + \|\tau - \tau_*\|_{l+1}) \tag{5.4.32}$$

for $l = 1, 2,\ u, v \in H_V^{l+1},\ \tau, \tau_* \in H_M^{l+1},\ \|u\|_{l+1} + \|v\|_{l+1} < 1,\ \|\tau\|_{l+1} + \|\tau_*\|_{l+1} < 1$.

Now we are going to use Theorem 3.2.1 twice. First, take

$$\begin{aligned} E &= H_V^1 \times H_M^1,\ v_0 = (a, \tau_0),\\ f_1(t,(u,\tau)) &= (f(t) + \widetilde{F}(u) + N_1(\tau), F(u,\tau) + N_2(u) + G(u,\tau)),\\ \mathcal{B}(u,\tau) &= (B_0 u, B_0 \tau),\ D(\mathcal{B}) = H_V^3 \times H_M^3, \qquad (5.4.33)\\ \mathcal{A}(u,\tau)(v_1,v_2) &= (A(u)v_1, A_\varepsilon v_2), \mathcal{A}_0(v_1,v_2) = (A_0 v_1, A_\varepsilon v_2),\\ \alpha &= \frac{7}{8}, \quad \beta = 1, \quad R \text{ small enough.} \end{aligned}$$

Then estimate (3.2.1) follows from (5.4.9) with $l = 1$, estimate (3.2.3) follows from (5.4.19) and (5.4.20) with $l = 1$, and estimate (3.2.4) follows from C^1-smoothness of f and estimates (5.4.31), (5.4.32), (5.4.7) with $l = 1$.

Thus, the conditions of Theorem 3.2.1, a) hold. Hence, for K_7 small enough, system (5.3.16), (5.4.2), (5.3.18) has a solution (u,τ) in class (5.4.24) on some interval $[0, t_0]$. Moreover, by Theorem 3.1.5, formula (3.1.29), the function

$$\mathcal{A}_0^{\delta_1} \mathcal{A}(u,\tau)(u,\tau) : [0, t_0] \to E$$

is continuous for some $\delta_1 > 0$. But by Theorem 3.1.2 the operator

$$\mathcal{B}^{1-\delta_1+\delta} [\mathcal{A}(u,\tau)]^{-1+\delta_1} \mathcal{A}_0^{-\delta_1}$$

is bounded uniformly with respect to t for $0 < \delta < \delta_1$. This yields the estimate

$$\|\mathcal{B}^{\frac{1}{2}-\delta_1+\delta} [\mathcal{A}(u,\tau)]^{\delta_1} (u,\tau)\|_2 < C, \quad t \in [0, t_0] \qquad (5.4.34)$$

where C does not depend on t.

Now, let us apply Theorem 3.2.1 with

$$\begin{aligned} E &= H_V^2 \times H_M^2,\ D(\mathcal{B}) = H_V^4 \times H_M^4,\\ \alpha &= \frac{1}{2},\ \beta = \frac{1}{2} + \delta_2 \quad (0 < \delta_2 < \delta), \end{aligned}$$

and with $v_0,\ f_1,\ \mathcal{A},\ \mathcal{B}, R$ as in (5.4.33). Estimates (5.4.9), (5.4.19), (5.4.20), (5.4.31), (5.4.32), (5.4.7) with $l = 2$ imply again that the conditions of Theorem 3.2.1, a) hold. Hence, problem (5.3.16), (5.4.2), (5.3.18) has a solution in class (5.4.3) on some interval $[0, t_*]$. Furthermore, estimate (5.4.28) with κ small enough enables to apply Lemma 5.4.4. Therefore this solution is unique in class (5.4.24) on $[0, t_*]$, i.e. it coincides with (u,τ). But by Theorem 3.1.2 the operator

$$\mathcal{B}^{\beta} [\mathcal{A}(u,\tau)]^{-\delta_1} \mathcal{B}^{-\frac{1}{2}+\delta_1-\delta}$$

is uniformly (with respect to t) bounded in $E = H_V^2 \times H_M^2$. Then estimate (5.4.34) implies

$$\|\mathcal{B}^{\beta}(u,\tau)(t)\|_{H_V^2 \times H_M^2} < K_{23},\ t \in [0, t_*]$$

where K_{23} does not depend on t_* and t. Furthermore, by estimate (5.4.28) this solution is a priori bounded by a constant independent on t, t_*, t_0:

$$\|\mathcal{B}^{1/2}(u,\tau)(t)\|_{H_V^2 \times H_M^2} \le K_{24}, \tag{5.4.35}$$

and K_{24} is arbitrary small provided K_7 is small. By Theorem 3.2.1, b)

$$t_* = t_0,$$

so the solution (u,τ) belongs to class (5.4.3) on the segment $[0,t_0]$. Rewrite (5.4.35) as

$$\|\mathcal{B}(u,\tau)(t)\|_{H_V^1 \times H_M^1} \le K_{24}. \tag{5.4.36}$$

If K_{24} is sufficiently small, (5.4.36) yields

$$\|\mathcal{B}^{\frac{7}{8}}(u,\tau)(t)\|_{H_V^1 \times H_M^1} < K_{25} < R. \tag{5.4.37}$$

Thus, by Theorem 3.2.1, b) applied to (5.4.33), t_0 is equal to T.

5.5 Proof of the main theorems

5.5.1 Proof of Theorem 5.3.1

Evidently, there exist sequences

$$\begin{aligned}
&a_m \underset{m\to\infty}{\to} a \text{ in } H_V^3,\ a_m \in H_V^4,\\
&\tau_{0m} \underset{m\to\infty}{\to} \tau_0 \text{ in } H_M^3,\ \tau_{0m} \in H_M^4,\\
&f_m \underset{m\to\infty}{\to} f \text{ in } L_1(0,T;H_V^3),\ f_m \in L_1(0,T;H_V^3) \cap C^1([0,T];H_V^2).
\end{aligned}$$

Without loss of generality all triplets (a_m, τ_{0m}, f_m) satisfy estimate (5.3.19). Consider problems (5.3.16), (5.4.2), (5.3.18) with the data a_m, τ_{0m}, f_m and $\varepsilon = \frac{1}{m}$ for all natural m. If K_6 is sufficiently small, by Theorem 5.4.1 each of these problems has a unique solution (u_m, τ_m) in class (5.4.3) and all these solutions are bounded uniformly with respect to m by estimates (5.4.28), (5.4.29). Therefore without loss of generality (see Remark 2.1.1) we may assume that there exists a pair (u_*, τ_*) such that

$$\begin{aligned}
&u_m \to u_* \text{ in } L_\infty(0,T;H_V^3)\ * - \text{weakly},\\
&\tau_m \to \tau_* \text{ in } L_\infty(0,T;H_M^3)\ * - \text{weakly},\\
&u_m \to u_* \text{ in } L_2(0,T;H_V^4) \text{ weakly}.
\end{aligned}$$

Let us show that the sequence (u_m, τ_m) is fundamental in $C([0,T];H_V^2 \times H_M^2)$.

Let $w_{ij} = u_i - u_j$, $\sigma_{ij} = \tau_i - \tau_j$. Apply the same procedure as in the proof of Lemma 5.4.4: substitute (u_i, τ_i) into (5.3.16), (5.4.2) with $\varepsilon = \frac{1}{i}$ and $(u_j,\ \tau_j)$ into (5.3.16), (5.4.2) with $\varepsilon = \frac{1}{j}$, take the differences of the corresponding equations, take the scalar product of the first of the obtained equations with $w_{ij}(t)$ in H^2_V and of the second one with $\frac{\sigma_{ij}}{2\eta}$ in H^2_M at every $t \in [0, T]$, and taking into account (5.4.27) add the obtained equalities :

$$\begin{aligned}
&\Big(\frac{dw_{ij}}{dt}, w_{ij}\Big)_2 + \frac{1}{2\eta}\Big(\frac{d\sigma_{ij}}{dt}, \sigma_{ij}\Big)_2\\
&\quad= -(A(u_i)u_i - A(u_j)u_j, B_0 w_{ij})_1 + (w_{ij}, w_{ij})_2 + (F_1(w_{ij}, u_i), w_{ij})_2\\
&\qquad + (F_1(u_j, w_{ij}), w_{ij})_2 - \frac{1}{2\eta\lambda}(\sigma_{ij}, \sigma_{ij})_2 + \frac{1}{2\eta}\Big(\frac{\Delta\tau_i}{i} - \frac{\Delta\tau_j}{j}, \sigma_{ij}\Big)_2\\
&\qquad + (F(w_{ij}, \tau_i), \sigma_{ij})_2 + (F(u_j, \sigma_{ij}), \sigma_{ij})_2 + (G(u_i, \tau_i)\\
&\qquad - G(u_j, \tau_j), \sigma_{ij})_2 + (f_i - f_j, w_{ij})_2.
\end{aligned}$$

Let K_6 be small enough. Using the estimates and arguing as in the proof of Lemma 5.4.4 we conclude:

$$\begin{aligned}
&\frac{d}{dt}\|w_{ij}\|_2^2 + \frac{1}{2\eta}\frac{d}{dt}\|\sigma_{ij}\|_2^2\\
&\quad\le C\big(\|w_{ij}\|_2^2 + \frac{1}{2\eta}\|\sigma_{ij}\|_2^2 + \|f_i - f_j\|_2\big) + \frac{1}{i\eta}\|\tau_i\|_3\|\sigma_{ij}\|_3 + \frac{1}{j\eta}\|\tau_j\|_3\|\sigma_{ij}\|_3\\
&\quad\le C\big(\|w_{ij}\|_2^2 + \frac{1}{2\eta}\|\sigma_{ij}\|_2^2 + \|f_i - f_j\|_2 + \frac{1}{i} + \frac{1}{j}\big).
\end{aligned}$$

Integrate from 0 to t:

$$\begin{aligned}
&\|w_{ij}\|_2^2 + \frac{1}{2\eta}\|\sigma_{ij}\|_2^2\\
&\qquad\le \|a_i - a_j\|_2^2 + \frac{1}{2\eta}\|\tau_{0i} - \tau_{0j}\|_2^2\\
&\qquad\quad + C\int_0^t \big(\|w_{ij}\|_2^2(s) + \frac{1}{2\eta}\|\sigma_{ij}\|_2^2(s) + \|f_i - f_j\|_2(s) + \frac{1}{i} + \frac{1}{j}\big)ds
\end{aligned}$$

and by Gronwall's lemma $\|w_{ij}\|_2^2(t) + \frac{1}{2\eta}\|\sigma_{ij}\|_2^2(t) \underset{\min(i,j)\to\infty}{\longrightarrow} 0$ uniformly with respect to $t \in [0, T]$.

Thus,

$$\begin{aligned}
u_m &\to u_* \text{ in } C([0, T]; H^2_V),\\
\tau_m &\to \tau_* \text{ in } C([0, T]; H^2_M).
\end{aligned}$$

This implies

$$\frac{1}{m}\Delta\tau_m \to 0 \text{ in } C([0,T];H_M^0),$$
$$A_{\frac{1}{m}}\tau_m \to \frac{\tau_*}{\lambda} \text{ in } C([0,T];H_M^0),$$
$$N_1(\tau_m) \to N_1(\tau_*) \text{ in } C([0,T];H_V^1),$$

$$N_2(u_m) \to N_2(u_*) \text{ in } C([0,T];H_M^1),$$
$$\frac{du_m}{dt} \to \frac{du_*}{dt} \text{ in the sense of distributions,}$$
$$\frac{d\tau_m}{dt} \to \frac{d\tau_*}{dt} \text{ in the sense of distributions.}$$

From estimates (5.4.4), (5.4.5), (5.4.7) we have also

$$\widetilde{F}(u_m) \to \widetilde{F}(u_*) \text{ in } C([0,T];H_V^1),$$
$$F(u_m,\tau_m) \to F(u_*,\tau_*) \text{ in } C([0,T];H_M^1),$$
$$G(u_m,\tau_m) \to G(u_*,\tau_*) \text{ in } C([0,T];H_M^1).$$

Furthermore, using (5.4.9), (5.4.10) we obtain

$$\begin{aligned}
&\operatorname{vrai}\max_{t\in[0,T]} \|A(u_m)u_m - A(u_*)u_*\|_0(t)\\
&\quad\le \operatorname{vrai}\max_{t\in[0,T]} \big(\|A(0)(u_m-u_*)(t)\|_0 + \|(A(u_*)-A(0))(u_m-u_*)\|_0(t)\\
&\qquad + \|(A(u_m)-A(u_*))u_m\|_0(t)\big)\\
&\quad\le C\operatorname{vrai}\max_{t\in[0,T]} \big(\|u_m-u_*\|_2 + \|u_*\|_{\frac{11}{4}}\|u_m-u_*\|_2 + \|u_m-u_*\|_{\frac{3}{2}}\|u_m\|_3\big).
\end{aligned}$$

Therefore

$$A(u_m)u_m \to A(u_*)u_* \text{ in } L_\infty(0,T;H_V^0).$$

Passing to the limit we conclude that the pair (u_*,τ_*) is a solution of problem (5.3.16) – (5.3.18).

Remark 5.5.1. Substituting u_m, f_m, τ_m into (5.3.16) and (5.3.17) we observe that the sequences $\frac{du_m}{dt}$, $\frac{d\tau_m}{dt}$ are bounded (and fundamental) in $L_1(0,T;H_V^0)$ and in $C([0,T];H_M^1)$, respectively. But $u_m \to u_*$ in $L_\infty(0,T;H_V^3)$ $*-$ weakly, and $\{u_m\}$ is a bounded sequence in $L_2(0,T;H_V^4)$. Since the embedding $H^4(B)\subset H^3(B)$ is compact for any ball B in $\mathbb{R}^n$, due to Theorem 2.2.7 without loss of generality we may assume that $u_m\big|_B \to u_*\big|_B$ in $L_p(0,T;H^3(B,\mathbb{R}^n))$ strongly for all $1\le p<\infty$ and every ball B. Furthermore, the embedding $H^3(B)\subset H^{3-\delta}(B)$ is compact for any $\delta > 0$, and $\tau_m \to \tau_*$ in $L_\infty(0, T; H^3_M)$ $*-$ weakly, so by Theorem 2.2.6 without loss of generality we may assume that $\tau_m\big|_B \to \tau_*\big|_B$ in $C([0,T]; H^{3-\delta}(B, \mathbb{R}^{n\times n}_S))$ strongly. These facts will be used in Section 5.6.

Let us return to the proof of the theorem. Estimate (5.4.9) yields that

$$\int_0^T \|A(u_*)u_*\|_2^2 \le 2\int_0^T \|(A(u_*) - A(0))u_*\|_2^2 + 2\int_0^T \|A(0)u_*\|_2^2$$
$$\le C\int_0^T (\|u_*\|_3^2\|u_*\|_4^2 + \|u_*\|_4^2).$$

Therefore $A(u_*)u_*$ belongs to $L_2(0,T; H^2_V)$. From estimate (5.4.4) we have $\widetilde{F}(u_*) \in L_\infty(0,T; H^2_V)$. Note that $N_1(\tau_*) \in L_\infty(0,T; H^2_V)$ and $f \in L_2(0,T; H^2_V)$. Substituting (u_*, τ_*) into (5.3.16), we conclude that $\frac{du_*}{dt} \in L_2(0,T; H^2_V)$. But $u_* \in L_2(0,T; H^4_V)$ and by Lemma 2.2.7 we have $u_* \in C([0,T]; H^3_V)$.

Substituting u_*, τ_* into (5.3.17) and taking into account estimates (5.4.5) and (5.4.7) we conclude that all terms in (5.3.17) except $\frac{d\tau_*}{dt}$ belong to $C([0,T]; H^1_M)$. Hence, $\tau_* \in C^1([0,T]; H^1_M)$.

The uniqueness of the solution may be proved in exactly the same way as Lemma 5.4.4.

5.5.2 Proof of Theorem 5.2.1

Estimates (5.4.7), (5.4.8) in the particular case $G(u,\tau) = \Phi(\mathcal{E}(u))$ have the form

$$\|\Phi(\mathcal{E}(u_1)) - \Phi(\mathcal{E}(u_2))\|_l \le K_{26}\cdot(\|u_1\|_{\gamma+1} + \|u_2\|_{\gamma+1})\|u_1 - u_2\|_{l+1}, \quad (5.5.1)$$

$$\|\Phi(\mathcal{E}(u))\|_3 \le K_{27}\cdot\|u\|_3\|u\|_4 \quad (5.5.2)$$

where K_{26}, K_{27} depend continuously on $\|\nabla u_1\|_\gamma + \|\nabla u_2\|_\gamma$ and on $\|u\|_3$, respectively, and l,γ are as in estimate (5.4.7).

Let $\tau_0 = \tau_0^1$, $K_5 = K_6$. Then (5.2.16) implies estimate (5.3.19). Hence, by Theorem 5.3.1 there exists a unique solution (u,τ) of problem (5.3.16) – (5.3.18) in class (5.3.20), (5.3.21). Denote

$$g = \frac{\partial u}{\partial t} + \sum_{i=1}^n u_i\frac{\partial u}{\partial x_i} - \eta_0\Delta u - \mathrm{Div}\,(\Phi(\mathcal{E}) + \tau) - f_0.$$

Taking into account notations (5.3.5) – (5.3.15) we conclude (cf. Remark 5.3.2) that (5.3.4) is valid and

$$Pg = 0.$$

Since P is an orthogonal projector in $H^0(\mathbb{R}^n, \mathbb{R}^n)$, g satisfies condition (3.1.3) with $\Omega = \mathbb{R}^n$.

Since u belongs to class (5.2.17), estimates (5.5.1), (5.5.2), and Lemma 2.2.9 yield that $\Phi(\mathcal{E}(u))$ belongs to class (5.2.20). Thus we see that $g \in L_2(0, T; H^2)$.

By Corollary 3.1.4 there exists unique p from class (5.2.19) satisfying (5.2.9) such that

$$\operatorname{grad} p(t) = -g(t) \text{ almost everywhere in } (0, T).$$

Therefore we have (5.3.3).

Let

$$\tau^1 = \tau, \quad \tau^0 = 2\eta_0 \mathcal{E}(u) + \Phi(\mathcal{E}(u)), \quad \mathrm{T} = -pI + \tau^0 + \tau^1.$$

Obviously, the triplet (u, T, p) is a solution of problem (5.2.1) – (5.2.10); τ^0 belongs to class (5.2.20), and T belongs to (5.2.18).

5.5.3 The case $r > 1$

The case $r > 1$ is investigated exactly in the same way as the case $r = 1$. During the operator treatment, problem (5.3.16) – (5.3.18) is replaced by the problem

$$\frac{du}{dt} + A(u)u = \widetilde{F}(u) + N_1\Big(\sum_{i=1}^{r} \tau^i\Big) + f, \tag{5.5.3}$$

$$\frac{d\tau^i}{dt} + \frac{\tau^i}{\lambda} = F(u, \tau^i) + N_2(u) + G_i(u, \tau^i), \tag{5.5.4}$$

$$u(0) = a, \quad \tau^i(0) = \tau_0^i, \quad i = 1, \dots, r, \tag{5.5.5}$$

where

$$G_i(u, \tau) = -\big(\tau W - W\tau + \frac{\beta_i(\tau, \mathcal{E})}{\lambda}\big).$$

Then one considers the following system

$$\frac{d\tau^i}{dt} + A_\varepsilon \tau^i = F(u, \tau^i) + N_2(u) + G_i(u, \tau^i) \tag{5.5.6}$$

and the auxiliary problem (5.5.3), (5.5.6), (5.5.5). Just as in the case $r = 1$, the analogues of Theorem 5.4.1 and Lemmas 5.4.4, 5.4.5 are proved and the passage to the limit is carried out.

5.6 Continuous dependence of solutions on data

In this section, we study continuous dependence of solutions to the initial problem for the equations of motion of nonlinear viscoelastic medium on the initial data and the body force. For simplicity, we consider the case $r = 1$. Thus, let us deal with the operator problem (5.3.16) – (5.3.18).

Theorem 5.6.1. *Let triples* $(a_k, \tau_{0k}, f_k), k = 0, 1, 2, \ldots,$ *satisfy the conditions of Theorem 5.3.1, including estimate (5.3.19). Let* (u_k, τ_k) *be the corresponding solutions to problem (5.3.16) – (5.3.18). Let*

$$
\begin{aligned}
a_k &\to a_0 \textit{ in } H_V^3, \\
\tau_{0k} &\to \tau_{00} \textit{ in } H_M^3, \\
f_k &\to f_0 \textit{ in } L_1(0, T; H_V^3),
\end{aligned}
$$

as $k \to \infty$. *Then*

$$
\begin{aligned}
u_k &\to u_0 \textit{ in } C([0,T]; H_V^2) \textit{ and } \textit{ in } L_p(0, T; H_{V,\mathrm{loc}}^3), \\
\tau_k &\to \tau_0 \textit{ in } C([0,T]; H_M^2) \textit{ and } \textit{ in } C([0,T]; H_{M,\mathrm{loc}}^{3-\delta}),
\end{aligned}
$$

for all $1 \le p < \infty, \delta > 0$.

Proof. Take some sequences

$$
\begin{aligned}
&a_{m,k} \underset{m\to\infty}{\to} a_k \text{ in } H_V^3, \ a_{m,k} \in H_V^4, \\
&\tau_{0m,k} \underset{m\to\infty}{\to} \tau_{0k} \text{ in } H_M^3, \ \tau_{0m,k} \in H_M^4, \\
&f_{m,k} \underset{m\to\infty}{\to} f_k \text{ in } L_1(0, T; H_V^3), \ f_{m,k} \in L_1(0, T; H_V^3) \cap C^1([0,T]; H_V^2).
\end{aligned}
$$

Without loss of generality the triples $(a_{m,k}, \tau_{0m,k}, f_{m,k})$ satisfy estimate (5.3.19). Consider problems (5.3.16), (5.4.2), (5.3.18) with data $a_{m,k}, \ \tau_{0m,k}, \ f_{m,k}$ and $\varepsilon = \frac{1}{m}$ for every natural m and for all k. If K_6 is sufficiently small, by Theorem 5.4.1 each of these problems possesses a unique solution $(u_{m,k}, \tau_{m,k})$. As is shown in the proof of Theorem 5.3.1,

$$
\begin{aligned}
&u_{m,k} \to u_k \text{ in } C([0,T]; H_V^2), u_{m,k}\big|_B \to u_k\big|_B \text{ in } L_p(0, T; H^3(B)), \\
&\tau_{m,k} \to \tau_k \text{ in } C([0,T]; H_M^2), \tau_{m,k}\big|_B \to \tau_k\big|_B \text{ in } C([0,T]; H^{3-\delta}(B)),
\end{aligned}
$$

as $m \to \infty$ for all $1 \le p < \infty, \delta > 0$ and for any ball B in $\mathbb{R}^n$.

Fix an arbitrary ball B in $\mathbb{R}^n$. Denote by

$$
\rho(u^1, \tau^1; u^2, \tau^2)
$$

the expression

$$
\begin{aligned}
&\|u^1 - u^2\|_{C([0,T];H_V^2)} + \|(u^1 - u^2)\big|_B\|_{L_p(0,T;H^3(B))} \\
&\quad + \|\tau^1 - \tau^2\|_{C([0,T];H_M^2)} + \|(\tau^1 - \tau^2)\big|_B\|_{C([0,T];H^{3-\delta}(B))},
\end{aligned}
$$

and denote by

$$\rho(a^1, \tau_0^1, f^1; a^2, \tau_0^2, f^2)$$

the expression

$$\|a^1 - a^2\|_{H_V^3} + \|\tau_0^1 - \tau_0^2\|_{H_M^3} + \|f^1 - f^2\|_{L_1(0,T;H_V^3)}$$

for any $u^1, \tau^1, u^2, \tau^2, a^1, a^2, \tau_0^1, \tau_0^2, f^1, f^2$ for which these expressions make sense.

Thus it suffices to prove that

$$\rho(u_k, \tau_k; u_0, \tau_0) \underset{k\to\infty}{\to} 0. \tag{5.6.1}$$

Fix $\varepsilon > 0$. For every natural k there is a number $L(k)$ such that if $m \geq L(k)$, then

$$\rho(u_{m,k}, \tau_{m,k}; u_k, \tau_k) \leq \frac{\varepsilon}{2}. \tag{5.6.2}$$

Besides, for any k there is $M(k)$ such that if $m \geq M(k)$, then

$$\rho(a_{m,k}, \tau_{0m,k}, f_{m,k}; a_k, \tau_{0k}, f_k) \leq \frac{1}{k}.$$

Let $P(k) = \max(L(k), M(k))$. Then

$$\begin{aligned}
&\rho(a_{P(k),k}, \tau_{0P(k),k}, f_{P(k),k}; a_0, \tau_{00}, f_0)\\
&\qquad \leq \rho(a_{P(k),k}, \tau_{0P(k),k}, f_{P(k),k}; a_k, \tau_{0k}, f_k) + \rho(a_k, \tau_{0k}, f_k; a_0, \tau_{00}, f_0)\\
&\qquad \leq \frac{1}{k} + \rho(a_k, \tau_{0k}, f_k; a_0, \tau_{00}, f_0) \underset{k\to\infty}{\to} 0.
\end{aligned}$$

As is shown in the proof of Theorem 5.3.1, this implies

$$\rho(u_{P(k),k}, \tau_{P(k),k}; u_0, \tau_0) \underset{k\to\infty}{\to} 0.$$

Then there is N such that if $k > N$, then

$$\rho(u_{P(k),k}, \tau_{P(k),k}; u_0, \tau_0) \leq \frac{\varepsilon}{2}.$$

Taking into account (5.6.2) and triangle inequality, we conclude

$$\rho(u_k, \tau_k; u_0, \tau_0) \leq \varepsilon,$$

so we have (5.6.1). □

Chapter 6

Weak solutions for equations of motion of viscoelastic medium

6.1 Preliminaries

6.1.1 Weak solutions for equations of fluid dynamics: general scheme

The feature of the most part of equations in fluid dynamics is that in the general case (i.e. without restrictions on the domain Ω and its dimension, the body force, and the initial data) the problem of existence of strong solutions to the initial-boundary value problems for these equations is open. A possible way to break this deadlock is to investigate generalized, *weak* solutions to these problems. There is a great variety of approaches to weak formulation of the problems of fluid mechanics (mostly of the Navier–Stokes problem; for instance, weak and generalized solutions in many senses, including the *Leray–Hopf* solutions, very weak and mild solutions, variational inequalities, see e.g. [6, 61, 37, 36]). Therefore, let us first describe a general scheme of weak problem formulation (not only for the problems of fluid dynamics), which includes most of the approaches.

Let $\mathcal{U}, \mathcal{P}, \mathcal{Z}$ be some sets, let $\mathcal{X} = \mathcal{U} \times \mathcal{P}$, and let $\mathcal{G} : \mathcal{X} \to \mathcal{Z}$ be a map. Denote by $\mathcal{J}$ the projection

$$\mathcal{J} : \mathcal{X} \to \mathcal{U}, \; \mathcal{J}(U, P) = U.$$

Consider an abstract equation

$$\mathcal{G}(U, P) = Z. \tag{6.1.1}$$

Here $U \in \mathcal{U}$ and $P \in \mathcal{P}$ are the unknowns, and $Z \in \mathcal{Z}_0 \subset \mathcal{Z}$ is prescribed ($\mathcal{Z}_0$ is some subset of $\mathcal{Z}$ containing the possible "right-hand sides" of the equation under consideration). Denote by $\mathcal{O}_s(Z) \subset \mathcal{X}$ the set of solutions to (6.1.1).

Remark 6.1.1. The set $\mathcal{P}$ can be one-point. Then (6.1.1) can be abbreviated:

$$\mathcal{G}_*(U) = Z \tag{6.1.2}$$

where $\mathcal{G}_*(U) = \mathcal{G}(U, P)$, $P \in \mathcal{P}$.

Remark 6.1.2. It is clear that Cauchy problems, boundary value problems, and initial-boundary value problems for partial differential equations can be expressed in form (6.1.1).

Let $\mathfrak{W}$ be some set containing $\mathfrak{U}$, let $\mathfrak{L}$ be some set containing $\mathfrak{Z}_0$, let $\mathfrak{M}$ be a fixed subset of some set $\mathfrak{Q}$, and let $\mathfrak{N} : \mathfrak{W} \times \mathfrak{L} \multimap \mathfrak{Q}$ be a multi-valued map. Consider the inclusion

$$\mathfrak{N}(W, L) \subset \mathfrak{M}. \tag{6.1.3}$$

Here $W \in \mathfrak{W}$ is unknown, and $L \in \mathfrak{L}$ is given. Denote by $\mathfrak{O}_w(L) \subset \mathfrak{W}$ the set of solutions to (6.1.3).

Assume that it can be *a priori* checked that for all $Z \in \mathfrak{Z}_0$ one has

$$\mathcal{J}(\mathfrak{O}_s(Z)) = \mathfrak{O}_w(Z) \bigcap \mathfrak{U}. \tag{6.1.4}$$

Then (6.1.3) is called the *weak statement* of (6.1.1), and the solutions to (6.1.3) are called *weak solutions* of (6.1.1).

Remark 6.1.3. Note that (6.1.3) may be studied not only for $L \in \mathfrak{Z}_0$ but for arbitrary L from $\mathfrak{L}$.

Let us illustrate this scheme with the weak formulation of the initial-boundary value problem for the Navier–Stokes system (1.1.14), (1.1.10), (1.1.15):

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} - \eta \Delta u + \operatorname{grad} p = f, \tag{6.1.5}$$

$$\operatorname{div} u = 0, \tag{6.1.6}$$

$$u\big|_{\partial\Omega} = 0, \tag{6.1.7}$$

$$u|_{t=0} = a. \tag{6.1.8}$$

Here u is an unknown velocity vector, p is an unknown pressure function, f is the given body force (all of them depend on a point x in a domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, and on a moment of time t), $\eta > 0$ is the viscosity, a is a given function of $x \in \Omega$.

Let $T > 0$ be a fixed moment of time. Denote $H_V^2 = H^2(\Omega)^n \cap V$.

Let

$$\begin{aligned}
\mathfrak{U} &= C^1([0, T]; H_V^2),\\
\mathcal{P} &= C([0, T]; H^1_{\mathrm{loc}}(\Omega)),\\
\mathfrak{Z} &= \mathfrak{Z}_0 = C([0, T]; L_2(\Omega)^n) \times H_V^2,\\
Z &= (f, a),
\end{aligned}$$

$$\mathfrak{G}(U, P) = \Big(\frac{\partial U}{\partial t} + \sum_{i=1}^{n} U_i \frac{\partial U}{\partial x_i} - \eta \Delta U + \operatorname{grad} P, U|_{t=0}\Big)$$

Then we can rewrite the Navier–Stokes problem in form (6.1.1). For the weak statement, take

$$\begin{aligned}\mathfrak{W} &= L_2(0,T;V)\bigcap C_w([0,T];H)\bigcap W_1^1(0,T;V^*),\\ \mathfrak{L} &= L_2(0,T;V^*)\times H,\\ \mathfrak{Q} &= L_1(0,T)\times H,\\ \mathfrak{M} &= \{(0 \text{ of } L_1(0,T), 0 \text{ of } H)\},\end{aligned}$$

and define the multi-valued map $\mathfrak{N}$ by the formula

$$\begin{aligned}\mathfrak{N}(W,L) = \Big\{\Big(&\frac{d}{dt}(W,\varphi)+\eta(\nabla W,\nabla\varphi)-\sum_{i=1}^{n}\big(W_iW,\frac{\partial\varphi}{\partial x_i}\big)\\ &-\langle f,\varphi\rangle_{V^*\times V}, W|_{t=0}-a\Big)\Big|\varphi\in V\Big\}\end{aligned}$$

for $W\in\mathfrak{W}$ and $L=(f,a)\in\mathfrak{L}$.

Then we arrive at the classical

Definition 6.1.1 (see e.g. [61]). A function

$$u\in L_2(0,T;V)\bigcap C_w([0,T];H)\bigcap W_1^1(0,T;V^*),$$

is a *weak solution* of problem (6.1.5) – (6.1.8) if it satisfies condition (6.1.8), and if the equality

$$\frac{d}{dt}(u,\varphi)+\eta(\nabla u,\nabla\varphi)-\sum_{i=1}^{n}\big(u_iu,\frac{\partial\varphi}{\partial x_i}\big)=\langle f,\varphi\rangle \tag{6.1.9}$$

is valid for all $\varphi\in V$ almost everywhere on $(0,T)$.

It remains to check (6.1.4). Let $Z=(f,a)\in\mathfrak{Z}_0=C([0,T];L_2(\Omega)^n)\times H_V^2$, and let $(u,p)\in\mathfrak{X}=\mathfrak{U}\times\mathcal{P}$ be a solution to (6.1.5) – (6.1.8). Taking the scalar product of (6.1.5) with an arbitrary function $\varphi\in V$ in $L_2(\Omega)^n$, we obtain

$$\frac{d}{dt}(u,\varphi)-\eta(\Delta u,\varphi)+\sum_{i=1}^{n}\big(u_i\frac{\partial u}{\partial x_i},\varphi\big)+(\operatorname{grad}p,\varphi)=(f,\varphi)=\langle f,\varphi\rangle. \tag{6.1.10}$$

The second equality follows from (2.2.27). Integrating by parts in the second, third, and forth terms (cf. Section 6.1.2), we arrive at

$$\frac{d}{dt}(u,\varphi)+\eta(\nabla u,\nabla\varphi)-\sum_{i=1}^{n}\big(u_iu,\frac{\partial\varphi}{\partial x_i}\big)-(p,\operatorname{div}\varphi)=\langle f,\varphi\rangle. \tag{6.1.11}$$

But $\operatorname{div}\varphi = 0$, so we obtain (6.1.9), and u is a weak solution to (6.1.5) – (6.1.8). Thus, $\mathcal{J}(\mathcal{O}_s(Z)) \subset \mathcal{O}_w(Z) \bigcap \mathcal{U}$.

Conversely, if $u \in \mathcal{U} = C^1([0,T]; H^2_V)$ is a weak solution, then integrating by parts in the second and the third terms of (6.1.9) we get

$$(u', \varphi) - \eta(\Delta u, \varphi) + \sum_{i=1}^{n} (u_i \frac{\partial u}{\partial x_i}, \varphi) - (f, \varphi) = 0 \tag{6.1.12}$$

for all $\varphi \in V$. Since $u' - \eta\Delta u + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} - f$ belongs to $C([0,T]; L_2)$, by Corollary 3.1.5 there is $q \in C([0,T]; H^1_{\mathrm{loc}}(\Omega))$ such that

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} - \eta\Delta u - f = \operatorname{grad} q(t). \tag{6.1.13}$$

Then the pair $(u, -q)$ belongs to $\mathcal{X}$ and is a solution to (6.1.5) – (6.1.8). Hence, $\mathcal{O}_w(Z) \bigcap \mathcal{U} \subset \mathcal{J}(\mathcal{O}_s(Z))$.

6.1.2 Integration by parts

As we see, integration by parts plays an important role in the analysis of weak solutions to the problems of fluid dynamics. Let us state a simple lemma about it.

Lemma 6.1.1. *Let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $u_j \in Z_j(\Omega)$ (where Z_j stands for some function space, e.g. W^r_p) for any $j = 1, \ldots, k$. Assume that the set of smooth functions on Ω is dense in each $Z_j(\Omega)$, and, moreover, $C^\infty_0(\Omega)$ is dense in $Z_m(\Omega)$ for some m, $1 \le m \le k$.*

Then, for $l = 1, \ldots, n$,

$$\sum_{j=1}^{k} \Big(\frac{\partial u_j}{\partial x_l}, u_1 u_2 \cdot \ldots \cdot u_{j-1} u_{j+1} \cdot \ldots \cdot u_k\Big) = 0 \tag{6.1.14}$$

provided the k-linear form in the left-hand side of (6.1.14) is continuous on $Z_1(\Omega) \times \ldots \times Z_k(\Omega)$.

Proof. Since the smooth functions are dense in $Z_j(\Omega)$, and $C^\infty_0(\Omega)$ is dense in $Z_m(\Omega)$, it remains to check (6.1.14) for smooth u_j, $j = 1, \ldots, k$, and $u_m \in C^\infty_0(\Omega)$.

But in this case the function $u_1 \cdot \ldots \cdot u_k$ has compact support in Ω, so

$$\begin{aligned}
\sum_{j=1}^{k} \Big(\frac{\partial u_j}{\partial x_l}, u_1 u_2 \cdot \ldots \cdot u_{j-1} u_{j+1} \cdot \ldots \cdot u_k\Big) \\
&= \sum_{j=1}^{k} \int_\Omega u_1 u_2 \cdot \ldots \cdot u_{j-1} \frac{\partial u_j}{\partial x_l} u_{j+1} \cdot \ldots \cdot u_k \, dx \\
&= \int_\Omega \frac{\partial}{\partial x_l} (u_1 u_2 \cdot \ldots \cdot u_k) \, dx = 0
\end{aligned}$$

by Green's formula. □

Corollary 6.1.1. *Formula (6.1.14) is valid in the following situations:*

a) *Ω is an arbitrary domain, $Z_j = W^1_{p_j}(\Omega)$ for $j = 1, \ldots, k$, $j \neq m$; $Z_m = \overset{\circ}{W}{}^1_{p_m}(\Omega)$ for some m; and*

$$\sum_{j=1}^{k} \frac{1}{p_j} = 1, \quad 1 \leq p_j \leq \infty \quad \Big(\frac{1}{\infty} = 0\Big). \tag{6.1.15}$$

b) *Ω is an arbitrary domain, $Z_j = \overset{\circ}{W}{}^1_{p_j}(\Omega)$ for every $j = 1, \ldots, k$, and there are $q_j \geq 0$ such that*

$$\frac{n}{p_j} - 1 \leq \frac{n}{q_j}, \quad 1 \leq p_j < \infty,$$
$$\frac{1}{p_j} + \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_{j-1}} + \frac{1}{q_{j+1}} + \cdots + \frac{1}{q_k} \leq 1.$$

c) *Ω is a sufficiently regular domain, $Z_j = W^1_{p_j}(\Omega)$ for $j = 1, \ldots, k$, $j \neq m$; $Z_m = \overset{\circ}{W}{}^1_{p_m}(\Omega)$ for some m; and for every $j = 1, \ldots, k$ there are $q_j \geq 0$ such that*

$$\frac{n}{p_j} - 1 \leq \frac{n}{q_j}, \quad 1 \leq p_j < \infty,$$
$$\frac{1}{p_j} + \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_{j-1}} + \frac{1}{q_{j+1}} + \cdots + \frac{1}{q_k} \leq 1.$$

Proof. We have to show that the k-linear form in the left-hand side of (6.1.14) is continuous on $Z_1(\Omega) \times \ldots \times Z_k(\Omega)$. It suffices to check that the forms $(\frac{\partial u_j}{\partial x_l}, u_1 u_2 \cdot$

$\dots \cdot u_{j-1}u_{j+1} \cdot \dots \cdot u_k)$ are bounded. In the case a), it's a direct consequence of Hölder's inequality (2.1.1):

$$
\begin{aligned}
&\Big(\frac{\partial u_j}{\partial x_l}, u_1 u_2 \cdot \dots \cdot u_{j-1} u_{j+1} \cdot \dots \cdot u_k\Big) \\
&\quad = \int\limits_{\Omega} \frac{\partial u_j}{\partial x_l} u_1 u_2 \cdot \dots \cdot u_{j-1} u_{j+1} \cdot \dots \cdot u_k \, dx \\
&\quad \le \Big\| \frac{\partial u_j}{\partial x_l} u_1 u_2 \cdot \dots \cdot u_{j-1} u_{j+1} \cdot \dots \cdot u_k \Big\|_{L_1(\Omega)} \\
&\quad \le \Big\| \frac{\partial u_j}{\partial x_l} \Big\|_{L_{p_j}(\Omega)} \|u_1\|_{L_{p_1}(\Omega)} \|u_2\|_{L_{p_2}(\Omega)} \cdot \dots \\
&\qquad \cdot \|u_{j-1}\|_{L_{p_{j-1}}(\Omega)} \|u_{j+1}\|_{L_{p_{j+1}}(\Omega)} \cdot \dots \cdot \|u_k\|_{L_{p_k}(\Omega)} \\
&\quad \le \|u_j\|_{W^1_{p_j}(\Omega)} \|u_1\|_{W^1_{p_1}(\Omega)} \|u_2\|_{W^1_{p_2}(\Omega)} \cdot \dots \\
&\qquad \cdot \|u_{j-1}\|_{W^1_{p_{j-1}}(\Omega)} \|u_{j+1}\|_{W^1_{p_{j+1}}(\Omega)} \cdot \dots \cdot \|u_k\|_{W^1_{p_k}(\Omega)}.
\end{aligned}
$$

In the cases b) and c), let us assume without loss of generality that $j \neq 1$. Let s be such that

$$
\frac{1}{p_j} + \frac{1}{s} + \frac{1}{q_2} + \dots + \frac{1}{q_{j-1}} + \frac{1}{q_{j+1}} + \dots + \frac{1}{q_k} = 1.
$$

It is easy to see that $1 \le s \le q_1$ and $\frac{n}{p_1} - 1 \le \frac{n}{s}$. By Theorem 2.1.1 a), we have continuous embeddings $Z_j(\Omega) \subset L_{q_j}$ and $Z_1(\Omega) \subset L_s$. Applying Hölder's inequality (2.1.1), we conclude:

$$
\begin{aligned}
&\Big(\frac{\partial u_j}{\partial x_l}, u_1 u_2 \cdot \dots \cdot u_{j-1} u_{j+1} \cdot \dots \cdot u_k\Big) \\
&\quad \le \Big\| \frac{\partial u_j}{\partial x_l} u_1 u_2 \cdot \dots \cdot u_{j-1} u_{j+1} \cdot \dots \cdot u_k \Big\|_{L_1(\Omega)} \\
&\quad \le \Big\| \frac{\partial u_j}{\partial x_l} \Big\|_{L_{p_j}(\Omega)} \|u_1\|_{L_s(\Omega)} \|u_2\|_{L_{q_2}(\Omega)} \cdot \dots \\
&\qquad \cdot \|u_{j-1}\|_{L_{q_{j-1}}(\Omega)} \|u_{j+1}\|_{L_{q_{j+1}}(\Omega)} \cdot \dots \cdot \|u_k\|_{L_{q_k}(\Omega)} \\
&\quad \le C \|u_j\|_{W^1_{p_j}(\Omega)} \|u_1\|_{Z_1(\Omega)} \|u_2\|_{Z_2(\Omega)} \cdot \dots \\
&\qquad \cdot \|u_{j-1}\|_{Z_{j-1}(\Omega)} \|u_{j+1}\|_{Z_{j+1}(\Omega)} \cdot \dots \cdot \|u_k\|_{Z_k(\Omega)} \\
&\quad = C \|u_1\|_{Z_1(\Omega)} \cdot \dots \cdot \|u_k\|_{Z_k(\Omega)}.
\end{aligned}
$$

□

Remark 6.1.4. Formula (6.1.14) may be rewritten as a formula of integration by parts:

$$\begin{aligned}
&\Big(\frac{\partial u_r}{\partial x_l}, u_1 u_2 \cdot \ldots \cdot u_{r-1} u_{r+1} \cdot \ldots \cdot u_k\Big) \\
&\quad = -\sum_{j=1}^{r-1} \Big(\frac{\partial u_j}{\partial x_l}, u_1 u_2 \cdot \ldots \cdot u_{j-1} u_{j+1} \cdot \ldots \cdot u_k\Big) \\
&\quad\quad - \sum_{j=r+1}^{k} \Big(\frac{\partial u_j}{\partial x_l}, u_1 u_2 \cdot \ldots \cdot u_{j-1} u_{j+1} \cdot \ldots \cdot u_k\Big)
\end{aligned} \tag{6.1.16}$$

for any $r = 1, \ldots, k$.

Corollary 6.1.2. *Let Ω be an arbitrary domain in $\mathbb{R}^n$. The following identities hold:*

$$(u, \operatorname{grad} q) = -(\operatorname{div} u, q), \tag{6.1.17}$$

$$\sum_{i=1}^{n} \Big(u_i u, \frac{\partial u}{\partial x_i}\Big) = 0, \tag{6.1.18}$$

$$\sum_{i=1}^{n} \Big(u_i \tau, \frac{\partial \tau}{\partial x_i}\Big) = 0, \tag{6.1.19}$$

$$(\tau, \nabla u) + (u, \operatorname{Div} \tau) = 0, \tag{6.1.20}$$

$$\sum_{i=1}^{n} \Bigg(\frac{u_i u}{1 + \delta\Big(\frac{|\tau|^2}{2\mu_2} + |u|^2\Big)}, \frac{\partial u}{\partial x_i}\Bigg) + \frac{1}{2\mu_2}\Bigg(\frac{u_i \tau}{1 + \delta\Big(\frac{|\tau|^2}{2\mu_2} + |u|^2\Big)}, \frac{\partial \tau}{\partial x_i}\Bigg) = 0. \tag{6.1.21}$$

In identities (6.1.17), (6.1.20) $u \in H_0^1(\Omega)^n, q \in H^1(\Omega)$, $\tau \in W_2^1(\Omega, \mathbb{R}_S^{n\times n})$. In identities (6.1.18), (6.1.19), (6.1.21) $u \in V$. In (6.1.19) $\tau \in W_4^1(\Omega, \mathbb{R}_S^{n\times n})$ or $H_0^1(\Omega, \mathbb{R}_S^{n\times n})$. In (6.1.21) $\tau \in W_2^1(\Omega, \mathbb{R}_S^{n\times n})$, $\mu_2, \delta > 0$. In identities (6.1.18), (6.1.19) $n \leq 4$.

Proof. Equality (6.1.17) is shown by a direct application of integration by parts (the case a) of Corollary 6.1.1):

$$\begin{aligned}
(u, \operatorname{grad} q) &= \sum_{i=1}^{n} \Big(u_i, \frac{\partial q}{\partial x_i}\Big) \\
&= -\sum_{i=1}^{n} \Big(\frac{\partial u_i}{\partial x_i}, q\Big) = -(\operatorname{div} u, q).
\end{aligned}$$

Similarly, we have

$$(\tau, \nabla u) = \sum_{i,j=1}^{n} \big(\tau_{ij}, \frac{\partial u_i}{\partial x_j}\big)$$
$$= -\sum_{i,j=1}^{n} \big(u_i, \frac{\partial \tau_{ij}}{\partial x_j}\big) = -(u, \operatorname{Div} \tau),$$

and this yields (6.1.20).

To show (6.1.21), we transform its left-hand side as follows

$$\sum_{i=1}^{n} \Big(\frac{u_i u}{1+\delta\Big(\frac{|\tau|^2}{2\mu_2}+|u|^2\Big)}, \frac{\partial u}{\partial x_i}\Big) + \frac{1}{2\mu_2}\Big(\frac{u_i \tau}{1+\delta\Big(\frac{|\tau|^2}{2\mu_2}+|u|^2\Big)}, \frac{\partial \tau}{\partial x_i}\Big)$$
$$= \sum_{i=1}^{n} \int_\Omega \frac{u_i}{1+\delta\Big(\frac{|\tau|^2}{2\mu_2}+|u|^2\Big)} \big[\big(u, \frac{\partial u}{\partial x_i}\big)_{\mathbb{R}^n} + \frac{1}{2\mu_2}\big(\tau, \frac{\partial \tau}{\partial x_i}\big)_{\mathbb{R}^{n\times n}_S}\big]\, dx$$
$$= \frac{1}{2}\sum_{i=1}^{n} \int_\Omega \frac{u_i}{1+\delta\Big(\frac{|\tau|^2}{2\mu_2}+|u|^2\Big)} \frac{\partial}{\partial x_i}\big(|u|^2 + \frac{1}{2\mu_2}|\tau|^2\big)\, dx$$
$$= \frac{1}{2\delta}\sum_{i=1}^{n} \int_\Omega u_i \frac{\partial}{\partial x_i} \ln\big(1+\delta\big(\frac{|\tau|^2}{2\mu_2}+|u|^2\big)\big)\, dx$$
$$= -\frac{1}{2\delta}\int_\Omega \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} \ln\big(1+\delta\big(\frac{|\tau|^2}{2\mu_2}+|u|^2\big)\big)\, dx$$
$$= -\frac{1}{2\delta}\int_\Omega \operatorname{div} u \cdot \ln\big(1+\delta\big(\frac{|\tau|^2}{2\mu_2}+|u|^2\big)\big)\, dx = 0.$$

We have applied integration by parts in the case a) of Corollary 6.1.1.

If $n \le 4$, by integration by parts, using case a) of Corollary 6.1.1 if $\tau \in W_4^1(\Omega, \mathbb{R}^{n\times n}_S)$ and case b) if $\tau \in H_0^1(\Omega, \mathbb{R}^{n\times n}_S)$, one has

$$\sum_{i=1}^{n} \big(u_i \tau, \frac{\partial \tau}{\partial x_i}\big) = -\sum_{i=1}^{n} \big(\frac{\partial u_i}{\partial x_i}, |\tau|^2\big) - \sum_{i=1}^{n} \big(\frac{\partial \tau}{\partial x_i}, u_i \tau\big)$$
$$= -(\operatorname{div} u, |\tau|^2) - \sum_{i=1}^{n} \big(u_i \tau, \frac{\partial \tau}{\partial x_i}\big) = -\sum_{i=1}^{n} \big(u_i \tau, \frac{\partial \tau}{\partial x_i}\big),$$

and we obtain (6.1.19). The proof of (6.1.18) is similar. □

6.2 Initial-boundary value problem for equations of motion of a viscoelastic medium with Jeffreys' constitutive law and its weak formulation

6.2.1 Statement of the problem

Let Ω be an arbitrary domain in the space $\mathbb{R}^n$, $n = 2, 3$, which, in particular, may be unbounded.

We consider the initial-boundary value problem which describes the motion of an incompressible viscoelastic medium with Jeffreys' constitutive law (1.3.12) which corresponds to the substantial derivative (1.3.7):

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} + \operatorname{grad} p = \operatorname{Div} \sigma + f, \tag{6.2.1}$$

$$\sigma + \lambda_1\Big(\frac{\partial \sigma}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial \sigma}{\partial x_i}\Big) = 2\eta\Big(\mathcal{E} + \lambda_2\Big(\frac{\partial \mathcal{E}}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial \mathcal{E}}{\partial x_i}\Big)\Big), \tag{6.2.2}$$

$$\operatorname{div} u = 0, \tag{6.2.3}$$

$$u\big|_{\partial\Omega} = 0, \tag{6.2.4}$$

$$u|_{t=0} = a, \ \sigma|_{t=0} = \sigma_0. \tag{6.2.5}$$

Here, as usual, u is an unknown velocity vector, p is an unknown pressure function, σ is an unknown deviatoric stress tensor, f is the given body force (all of them depend on a point x in the domain Ω, and on a moment of time t); $\mathcal{E}(u)$, $\mathcal{E}(u) = (\mathcal{E}_{ij}(u))$, $\mathcal{E}_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, is the strain velocity tensor, $\eta > 0$ is the viscosity of the medium, λ_1 is the relaxation time, λ_2 is the retardation time, $0 < \lambda_2 < \lambda_1$, a and σ_0 are given functions.

Equation (6.2.1) is the equation of motion (1.1.12). Equation (6.2.2) is the Jeffreys constitutive law (1.3.12). Equation (6.2.3) is the equation of continuity (1.1.10). Equation (6.2.4) is the no-slip condition (1.1.15). Equation (6.2.5) is simply an initial condition.

6.2.2 Weak formulation of the problem

First let us point out that C_0^∞ will stand only for $C_0^\infty(\Omega, \mathbb{R}_S^{n\times n})$ in Chapter 6.

Let, as in Section 6.1.1, $T > 0$ be a fixed moment of time. Following the general scheme of weak setting, let

$$\begin{aligned}
\mathfrak{U} &= C^1([0,T];H^2_V)\times C^1([0,T];H^1(\Omega,\mathbb{R}^{n\times n}_S)),\\
\mathcal{P} &= C([0,T];H^1_{\mathrm{loc}}(\Omega)),\\
\mathfrak{Z} &= C([0,T];L_2(\Omega)^n)\times C([0,T];L_2(\Omega,\mathbb{R}^{n\times n}_S))\times H^2_V\times H^1(\Omega,\mathbb{R}^{n\times n}_S),\\
\mathfrak{Z}_0 &= C([0,T];L_2(\Omega)^n)\times\{0\}\times H^2_V\times H^1(\Omega,\mathbb{R}^{n\times n}_S),\\
Z &= (f,0,a,\sigma_0),\\
\mathcal{G}(U,P) &= \mathcal{G}((u,\sigma),P)\\
&= \Big(\frac{\partial u}{\partial t}+\sum_{i=1}^n u_i\frac{\partial u}{\partial x_i}+\mathrm{grad}\,P-\mathrm{Div}\ \sigma,\\
&\qquad \sigma+\lambda_1\big(\frac{\partial\sigma}{\partial t}+\sum_{i=1}^n u_i\frac{\partial\sigma}{\partial x_i}\big)-2\eta\big(\mathcal{E}(u)+\lambda_2\big(\frac{\partial\mathcal{E}(u)}{\partial t}+\sum_{i=1}^n u_i\frac{\partial\mathcal{E}(u)}{\partial x_i}\big)\big),\\
&\qquad u|_{t=0},\sigma|_{t=0}\Big).
\end{aligned}$$

Then we can rewrite problem (6.2.1) – (6.2.5) in form (6.1.1). In order to define weak solutions, take

$$\begin{aligned}
\mathfrak{W} &= L_2(0,T;V)\bigcap C_w([0,T];H)\bigcap W^1_1(0,T;V^*)\\
&\qquad\times L_2(0,T;L_2(\Omega,\mathbb{R}^{n\times n}_S))\bigcap C_w([0,T];H^{-1}(\Omega,\mathbb{R}^{n\times n}_S)),\\
\mathfrak{L} &= L_2(0,T;V^*)\times\{0\}\times H\times W_2^{-1}(\Omega,\mathbb{R}^{n\times n}_S),\\
\mathfrak{Q} &= \mathcal{D}'(0,T)\times\mathcal{D}'(0,T)\times H\times W_2^{-1}(\Omega,\mathbb{R}^{n\times n}_S),\\
\mathfrak{M} &= \{(0,0,0,0)\},\\
\mathfrak{N}(W,L) &= \Big\{\Big(\frac{d}{dt}(u,\varphi)+(\sigma,\nabla\varphi)-\sum_{i=1}^n\big(u_iu,\frac{\partial\varphi}{\partial x_i}\big)-\langle f,\varphi\rangle,\\
&\qquad (\sigma,\Phi)+\lambda_1\frac{d}{dt}(\sigma,\Phi)-\lambda_1\sum_{i=1}^n\big(u_i\sigma,\ \frac{\partial\Phi}{\partial x_i}\big)\\
&\qquad\quad +2\eta(u,\mathrm{Div}\,\Phi)+2\eta\lambda_2\big(\frac{d}{dt}(u,\mathrm{Div}\,\Phi)+\sum_{i=1}^n\big(u_i\mathcal{E}(u),\frac{\partial\Phi}{\partial x_i}\big)\big),\\
&\qquad u|_{t=0}-a,\sigma|_{t=0}-\sigma_0\Big)\Big|\varphi\in\mathcal{V},\Phi\in C_0^\infty\Big\}
\end{aligned}$$

for $W=(u,\sigma)\in\mathfrak{W}$ and $L=(f,0,a,\sigma_0)\in\mathfrak{L}$.

Then we arrive at

Definition 6.2.1. A pair of functions (u, σ),

$$
\begin{aligned}
&u \in L_2(0,T;V) \bigcap C_w([0,T];H), \ \frac{du}{dt} \in L_1(0,T;V^*), \\
&\sigma \in L_2(0,T;L_2(\Omega,\mathbb{R}_S^{n\times n})) \bigcap C_w([0,T];H^{-1}(\Omega,\mathbb{R}_S^{n\times n}))
\end{aligned}
\tag{6.2.6}
$$

is a *weak solution* of problem (6.2.1) – (6.2.5) if it satisfies condition (6.2.5), and if the equalities

$$
\frac{d}{dt}(u,\varphi) + (\sigma, \nabla\varphi) - \sum_{i=1}^{n} \big(u_i u, \frac{\partial \varphi}{\partial x_i}\big) = \langle f, \varphi\rangle,
\tag{6.2.7}
$$

$$
\begin{aligned}
&(\sigma,\Phi) + \lambda_1 \frac{d}{dt}(\sigma,\Phi) - \lambda_1 \sum_{i=1}^{n} \big(u_i\sigma, \ \frac{\partial \Phi}{\partial x_i}\big) \\
&\quad = -2\eta(u, \operatorname{Div}\Phi) - 2\eta\lambda_2 \big(\frac{d}{dt}(u, \operatorname{Div}\Phi) + \sum_{i=1}^{n} \big(u_i \mathcal{E}(u), \frac{\partial \Phi}{\partial x_i}\big)\big)
\end{aligned}
\tag{6.2.8}
$$

are true for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$ in the sense of distributions on $(0,T)$ (i.e. in $\mathcal{D}'(0,T)$; see, however, Remark 6.5.3).

Condition (6.1.4) is checked just as for the Navier–Stokes problem (using integration by parts again).

6.2.3 An existence result

Now we are ready to formulate one of the main results of this chapter.

Theorem 6.2.1 (see [72]). *Given* $f \in L_2(0,T;V^*)$, $a \in H$, $\sigma_0 \in W_2^{-1}(\Omega,\mathbb{R}_S^{n\times n})$, $\sigma_0 - 2\eta\frac{\lambda_2}{\lambda_1}\mathcal{E}(a) \in L_2(\Omega,\mathbb{R}_S^{n\times n})$, *there exists a weak solution of problem (6.2.1) – (6.2.5) in class (6.2.6) .*

We shall prove this theorem in Section 6.5.

Remark 6.2.1. As a matter of fact one can show a bit more regularity than in (6.2.6) for the solution of problem (6.2.1) – (6.2.5) even under the conditions of Theorem 6.2.1. See Theorem 6.5.1 and Remark 6.8.1.

Remark 6.2.2. Existence of a weak solution for problem (6.2.1) – (6.2.5) (in a slightly different weak setting) was proved by Turganbaev [65] in the case when Ω is a bounded sufficiently regular domain and $f \in L_2(0,T;L_2)$.

Remark 6.2.3. In Chapters 6 and 7 it is possible to replace the space $\mathbb{R}_S^{n\times n}$ with $\mathbb{R}_D^{n\times n} = \{\xi \in \mathbb{R}_S^{n\times n} | \operatorname{Tr}\xi = 0\}$.

6.3 Auxiliary problem

Before proving Theorem 6.2.1, we study an auxiliary problem. Let us begin with an equivalent transformation of system (6.2.7), (6.2.8). Denote $\mu_1 = \eta\frac{\lambda_2}{\lambda_1}$, $\mu_2 = \frac{\eta-\mu_1}{\lambda_1}$, $\tau = \sigma - 2\mu_1\mathcal{E}(u)$. Then we can rewrite (6.2.8) and (6.2.7) as follows (cf. Section 1.5.1):

$$\frac{d}{dt}(\tau,\Phi) + \frac{1}{\lambda_1}(\tau,\Phi) - \sum_{i=1}^{n}(u_i\tau, \frac{\partial\Phi}{\partial x_i}) + 2\mu_2(u, \operatorname{Div}\Phi) = 0 \tag{6.3.1}$$

$$\frac{d}{dt}(u,\varphi) - \sum_{i=1}^{n}\big(u_i u, \frac{\partial\varphi}{\partial x_i}) + \mu_1(\nabla u, \nabla\varphi) + (\tau, \nabla\varphi) = \langle f, \varphi\rangle \tag{6.3.2}$$

for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$ (cf. Remark 6.5.1).

Remark 6.3.1. During the study of many issues concerning the weak solutions of problem (6.2.1) – (6.2.4) (see Sections 6.5 – 6.8) it appears to be convenient to pass to the variables (u,τ) and to investigate these issues for problem (6.3.1) – (6.3.2).

Consider the following auxiliary problem

$$\begin{aligned}\frac{d}{dt}(\tau,\Phi) + \frac{1}{\lambda_1}(\tau,\Phi) - \xi\sum_{i=1}^{n}\Big(\frac{u_i\tau}{1+\delta\Big(\frac{|\tau|^2}{2\mu_2}+|u|^2\Big)}, \frac{\partial\Phi}{\partial x_i}\Big)&\\ + 2\mu_2\xi(u,\operatorname{Div}\Phi) + \frac{\varepsilon}{\lambda_1}(\nabla\tau,\nabla\Phi) = 0&,\end{aligned} \tag{6.3.3}$$

$$\begin{aligned}\frac{d}{dt}(u,\varphi) - \xi\sum_{i=0}^{n}\Big(\frac{u_i u}{1+\delta\Big(\frac{|\tau|^2}{2\mu_2}+|u|^2\Big)}, \frac{\partial\varphi}{\partial x_i}\Big)&\\ + \mu_1(\nabla u,\nabla\varphi) + \xi(\tau,\nabla\varphi) = \langle f,\varphi\rangle&\end{aligned} \tag{6.3.4}$$

for all $\varphi \in V,\ \Phi \in H_0^1$ almost everywhere in $(0,T)$;

$$u|_{t=0} = a,\ \tau|_{t=0} = \tau_0. \tag{6.3.5}$$

The numbers $\delta > 0,\ 0 \le \xi \le 1, 0 < \varepsilon \le 1$ are parameters.

Let us introduce the following spaces,

$$W = \Big\{u \in L_2(0,T;V),\ \frac{du}{dt} \in L_2(0,T;V^*)\Big\},$$
$$\|u\|_W = \|u\|_{L_2(0,T;V)} + \|u'\|_{L_2(0,T;V^*)},$$

$$W_M = \{\tau \in L_2(0,T; H_0^1(\Omega, \mathbb{R}_S^{n\times n})),\ \frac{d\tau}{dt} \in L_2(0,T; H^{-1}(\Omega, \mathbb{R}_S^{n\times n}))\},$$
$$\|\tau\|_{W_M} = \|\tau\|_{L_2(0,T;H_0^1)} + \|\tau'\|_{L_2(0,T;H^{-1})}.$$

By Corollary 2.2.3, W and W_M are continuously embedded into $C([0,T];H)$ and $C([0,T];L_2)$, respectively. If Ω is bounded, the embeddings $V \subset H$ and $H_0^1 \subset L_2$ are compact, so the embeddings $W \subset L_2(0,T;H)$ and $W_M \subset L_2(0,T;L_2)$ are also compact by Theorem 2.2.6.

Lemma 6.3.1. *Let $a \in H$, $\tau_0 \in L_2$, $f \in L_2(0,T;V^*)$ and let a pair $(u \in W, \tau \in W_M)$ be a solution of problem (6.3.3) – (6.3.5). Then the following estimate holds,*

$$\max_{t\in[0,T]} \|u\|(t) + \max_{t\in[0,T]} \|\tau\|(t) + \int_0^T \|u\|_1^2(t)dt + \varepsilon \int_0^T \|\tau\|_1^2(t)dt \\ \leq K_0(\|a\|, \|\tau_0\|, \|f\|_{L_2(0,T;V^*)}) \tag{6.3.6}$$

where the constant K_0 does not depend on $\Omega, \varepsilon, \delta, \xi$.

Proof. It follows from Lemma 2.2.8 that

$$\langle\frac{du}{dt}, \varphi\rangle = \frac{d}{dt}(u,\varphi), \langle\frac{d\tau}{dt}, \Phi\rangle = \frac{d}{dt}(\tau, \Phi). \tag{6.3.7}$$

So, by (2.2.28),

$$\frac{d}{dt}(u,\varphi)\Big|_{\varphi=u(t)} = \langle\frac{du(t)}{dt}, u(t)\rangle = \frac{1}{2}\frac{d}{dt}(u(t),u(t)).$$

Analogously

$$\frac{d}{dt}(\tau,\Phi)\Big|_{\Phi=\tau(t)} = \frac{1}{2}\frac{d}{dt}(\tau,\tau).$$

Put $\Phi = \dfrac{\tau(t)}{2\mu_2}$ in (6.3.3) and $\varphi = u(t)$ in (6.3.4) for almost all $t \in [0,T]$, and add the results:

$$\frac{1}{2}\frac{d}{dt}(u,u) + \frac{1}{4\mu_2}\frac{d}{dt}(\tau,\tau) + \mu_1(\nabla u, \nabla u) + \frac{1}{2\lambda_1\mu_2}(\tau,\tau)$$
$$-\xi\sum_{i=1}^n\Big(\frac{u_i u}{1+\delta\Big(\frac{|\tau|^2}{2\mu_2}+|u|^2\Big)}, \frac{\partial u}{\partial x_i}\Big)$$
$$-\xi\frac{1}{2\mu_2}\Big(\frac{u_i\tau}{1+\delta\Big(\frac{|\tau|^2}{2\mu_2}+|u|^2\Big)}, \frac{\partial \tau}{\partial x_i}\Big) + \frac{\varepsilon}{2\lambda_1\mu_2}(\nabla\tau,\nabla\tau)$$
$$+\xi(\tau,\nabla u) + \xi(u, \operatorname{Div}\tau) = \langle f, u\rangle.$$

Taking into account (6.1.20) and (6.1.21), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(u,u) + \frac{1}{4\mu_2}\frac{d}{dt}(\tau,\tau) + \mu_1(\nabla u, \nabla u) + \frac{1}{2\lambda_1\mu_2}(\tau,\tau) \\
&\qquad + \frac{\varepsilon}{2\lambda_1\mu_2}(\nabla\tau,\nabla\tau) = \langle f, u\rangle.
\end{aligned} \tag{6.3.8}
$$

We are going to use the following simple inequality:

$$
\begin{aligned}
&\frac{1}{k}\big(\max_{s\in J}\phi_1(s) + \max_{s\in J}\phi_2(s) + \cdots + \max_{s\in J}\phi_k(s)\big) \\
&\qquad \le \max_{s\in J}(\phi_1(s) + \phi_2(s) + \cdots + \phi_k(s))
\end{aligned} \tag{6.3.9}
$$

for scalar functions $\phi_1, \phi_2, \ldots, \phi_k : J \subset \mathbb{R} \to [0, +\infty)$, $k \in \mathbb{N}$.

Integration of the terms in (6.3.8) from 0 to t yields:

$$
\begin{aligned}
&\frac{1}{2}\|u\|^2(t) + \frac{1}{4\mu_2}\|\tau\|^2(t) + \int_0^t \frac{1-\varepsilon}{2\lambda_1\mu_2}\|\tau\|^2\,ds + \int_0^t \frac{\varepsilon}{2\lambda_1\mu_2}\|\tau\|_1^2\,ds \\
&\quad + \int_0^t \mu_1(\|u\|_1^2 - \|u\|^2)\,ds \le \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \int_0^t \|f\|_{V^*}\,\|u\|_1\,ds.
\end{aligned}
$$

Then we get:

$$
\begin{aligned}
&\frac{1}{10}\max_{t\in[0,T]}\|u\|^2(t) + \frac{1}{20\mu_2}\max_{t\in[0,T]}\|\tau\|^2(t) + \int_0^T \frac{1-\varepsilon}{10\lambda_1\mu_2}\|\tau\|^2\,dt \\
&\qquad + \int_0^T \frac{\varepsilon}{10\lambda_1\mu_2}\|\tau\|_1^2\,dt + \frac{1}{5}\int_0^T \mu_1(\|u\|_1^2 - \|u\|^2)\,dt \\
&\quad \le \max_{t\in[0,T]}\Big[\frac{1}{2}\|u\|^2(t) + \frac{1}{4\mu_2}\|\tau\|^2(t) + \int_0^t \frac{1-\varepsilon}{2\lambda_1\mu_2}\|\tau\|^2\,ds \\
&\qquad + \int_0^t \frac{\varepsilon}{2\lambda_1\mu_2}\|\tau\|_1^2\,ds + \int_0^t \mu_1(\|u\|_1^2 - \|u\|^2)\,ds\Big] \\
&\quad \le \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \int_0^T \|f\|_{V^*}\,\|u\|_1\,dt \\
&\quad \le \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \|f\|_{L_2(0,T;V^*)}\Big(\int_0^T \|u\|_1^2\,dt\Big)^{1/2}.
\end{aligned} \tag{6.3.10}
$$

Note that the following inequality is valid:

$$
\max_{t\in[0,T]}\frac{1}{20}\|u\|^2(t) + \frac{1}{5}\int_0^T \mu_1(\|u\|_1^2 - \|u\|^2)\,dt \ge \gamma\int_0^T \|u\|_1^2\,dt \tag{6.3.11}
$$

where $\gamma = \min\left(\frac{1}{20T}, \frac{\mu_1}{5}\right)$.

For its proof it is enough to add inequalities:

$$\max_{t\in[0,T]} \frac{1}{20}\|u\|^2(t) \geq \frac{1}{20T}\int_0^T \|u\|^2\,dt \geq \gamma \int_0^T \|u\|^2\,dt$$

and

$$\frac{1}{5}\int_0^T \mu_1(\|u\|_1^2 - \|u\|^2)\,dt \geq \gamma \int_0^T (\|u\|_1^2 - \|u\|^2)\,dt.$$

Now, we have from (6.3.10) and (6.3.11):

$$\gamma \int_0^T \|u\|_1^2\,dt \leq \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \|f\|_{L_2(0,T;V^*)}\Big(\int_0^T \|u\|_1^2\,dt\Big)^{1/2}.$$

This yields $\left(\int_0^T \|u\|_1^2\,dt\right)^{1/2} \leq y_2$, where y_2 is the greater root of the quadratic equation

$$\gamma y^2 = \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \|f\|_{L_2(0,T;V^*)}\,y.$$

Then from (6.3.10) and (6.3.11) it follows that

$$\frac{1}{20}\max_{t\in[0,T]}\|u\|^2(t) + \frac{1}{20\mu_2}\max_{t\in[0,T]}\|\tau\|^2(t) + \gamma\int_0^T \|u\|_1^2\,dt + \frac{\varepsilon}{10\lambda_1\mu_2}\int_0^T \|\tau\|_1^2\,dt$$
$$\leq \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \|f\|_{L_2(0,T;V^*)}\,y_2,$$

what yields the statement of the lemma. □

Theorem 6.3.1. *Let Ω be bounded and a, τ_0, f satisfy the conditions of Lemma 6.3.1. Then problem (6.3.3) – (6.3.5) possesses a solution $u \in W$, $\tau \in W_M$.*

Proof. Let us introduce auxiliary operators by the following formulas (in these formulas φ and Φ are arbitrary elements of V and $H_0^1(\Omega, \mathbb{R}_S^{n\times n})$ respectively):

$$N_1 : W_M \to L_2(0,T;V^*),\ \langle N_1(\tau), \varphi\rangle = (\tau, \nabla\varphi)$$
$$N_2 : W \to L_2(0,T;H^{-1}),\ \langle N_2(u), \Phi\rangle = 2\mu_2(u, \operatorname{Div}\Phi)$$
$$K_\delta : W \times W_M \to L_2(0,T;V^*),$$
$$\langle K_\delta(u,\tau), \varphi\rangle = -\sum_{i=1}^n \left(\frac{u_i u}{1 + \delta\Big(|u|^2 + \frac{|\tau|^2}{2\mu_2}\Big)}, \frac{\partial\varphi}{\partial x_i}\right),$$

$$\tilde{K}_\delta : W \times W_M \to L_2(0,T;H^{-1}),$$

$$\langle \tilde{K}_\delta(u,\tau), \Phi\rangle = -\sum_{i=1}^{n}\Big(\frac{u_i\tau}{1+\delta\Big(|u|^2+\dfrac{|\tau|^2}{2\mu_2}\Big)}, \frac{\partial\Phi}{\partial x_i}\Big),$$

$$A : V \to V^*,\ \langle A(u),\varphi\rangle = \mu_1(\nabla u,\nabla\varphi),$$

$$A_\varepsilon : H_0^1 \to H^{-1},\ \langle A_\varepsilon(\tau),\Phi\rangle = \varepsilon(\nabla\tau,\nabla\Phi) + \frac{1}{\lambda_1}(\tau,\Phi),$$

$$\tilde{A} : W \times W_M \to L_2(0,T;V^*) \times L_2(0,T;H^{-1}) \times H \times L_2,$$

$$\tilde{A}(u,\tau) = \Big(\frac{du(t)}{dt} + A(u(t)), \frac{d\tau(t)}{dt} + A_\varepsilon(\tau(t)), u|_{t=0}, \tau|_{t=0}\Big),$$

$$Q : W \times W_M \to L_2(0,T;V^*) \times L_2(0,T;H^{-1}) \times H \times L_2,$$

$$Q(u,\tau) = (K_\delta(u,\tau) + N_1(\tau), \tilde{K}_\delta(u,\tau) + N_2(u), 0, 0).$$

Then problem (6.3.3) – (6.3.5) is equivalent to the operator equation

$$\tilde{A}(u,\tau) + \xi Q(u,\tau) = (f, 0, a, \tau_0). \tag{6.3.12}$$

Since the embeddings $W \subset L_2(0,T;H)$ and $W_M \subset L_2(0,T;L_2)$ are compact, the operators N_1, N_2 are compact. Let us show that the operator $\tilde{K}_\delta$ is also compact. This operator may be considered as a superposition of the embedding operator

$$j : W \times W_M \to L_2(0,T;H) \times L_2(0,T;L_2)$$

and of the operator

$$\tilde{K}_\delta : L_2(0,T;H) \times L_2(0,T;L_2) \to L_2(0,T;H^{-1}).$$

The first operator is compact, so it suffices to show that the second one is continuous. Observe that for this purpose it is enough to know that the following Nemytskii operators are continuous:

$$\phi_{ijk} : L_2((0,T)\times\Omega,\mathbb{R}^n) \times L_2((0,T)\times\Omega,\mathbb{R}^{n\times n}_S) \to L_2((0,T)\times\Omega),$$

$$\phi_{ijk}(v,\varsigma)(t,x) = \frac{v_i(t,x)\varsigma_{jk}(t,x)}{1+\delta\Big(|v(t,x)|^2 + \dfrac{|\varsigma(t,x)|^2}{2\mu_2}\Big)},\ i,j,k = 1,\dots,n.$$

But by Cauchy's inequality for all $v \in \mathbb{R}^n$ and $\varsigma \in \mathbb{R}^{n\times n}_S$ one has

$$\Big|\frac{v_i\varsigma_{jk}}{1+\delta\Big(|v|^2+\dfrac{|\varsigma|^2}{2\mu_2}\Big)}\Big| \le \Big|\frac{\frac{1}{2}(2\mu_2)^{1/2}v_i^2 + \frac{1}{2(2\mu_2)^{1/2}}\varsigma_{jk}^2}{1+\delta\Big(|v|^2+\dfrac{|\varsigma|^2}{2\mu_2}\Big)}\Big| \le \frac{(2\mu_2)^{1/2}}{2\delta}.$$

Therefore, by Krasnoselskii's theorem [31], the Nemytskii operators ϕ_{ijk} are continuous.

Similarly one checks that the operator K_δ is compact.

Hence, the operator Q is also compact. But the operator $\tilde{A}$ is continuously invertible by Lemma 3.1.3. Rewrite equation (6.3.12) as

$$(u,\tau) + \xi \tilde{A}^{-1} Q(u,\tau) = \tilde{A}^{-1}(f,0,a,\tau_0). \tag{6.3.13}$$

By Lemma 6.3.1 equation (6.3.13) has no solutions on the boundary of a sufficiently large ball B in $W \times W_M$, independent on ξ. Without loss of generality $a_0 = \tilde{A}^{-1}(f,0,a,\tau_0)$ belongs to this ball. Then we can consider the Leray–Schauder degree (see Section 3.2.2) of the map $I + \xi\tilde{A}^{-1}Q$ on the ball B with respect to the point a_0,

$$\deg_{LS}(I + \xi\tilde{A}^{-1}Q, B, a_0),$$

where I is the identity operator. By the homotopic invariance property of the degree we have

$$\deg_{LS}(I + \xi\tilde{A}^{-1}Q, B, a_0) = \deg_{LS}(I, B, a_0) = 1.$$

By Theorem 3.2.3, equation (6.3.13) (and therefore, problem (6.3.3) – (6.3.5)) has a solution in the ball B for every ξ. □

We need the following estimates on the time derivatives of the solutions of problem (6.3.3) – (6.3.5).

Lemma 6.3.2. *Under the conditions of the previous theorem the following estimates of the solutions are valid:*

$$\Big\|\frac{du}{dt}\Big\|_{L_1(0,T;V^*)} \le K_1\big(\|a\|,\ \|\tau_0\|, \|f\|_{L_2(0,T;V^*)}\big), \tag{6.3.14}$$

$$\Big\|\frac{d\tau}{dt}\Big\|_{L_1(0,T;H^{-1})} \le K_2\big(\|a\|, \|\tau_0\|, \|f\|_{L_2(0,T;V^*)}, \varepsilon\big) \tag{6.3.15}$$

where the constants K_1, K_2 do not depend on Ω, δ, ξ, and K_1 is independent of ε.

Proof. We have from (6.3.4):

$$\begin{aligned}\|\langle u',\varphi\rangle\|_{L_1(0,T)} &= \int_0^T \Big|\frac{d}{dt}(u,\varphi)\Big|dt \\ &\le \int_0^T \Big|\xi\sum_{i=1}^n\Big(\frac{u_i u}{1+\delta\Big(\frac{|\tau|^2}{2\mu_2}+|u|^2\Big)}, \frac{\partial\varphi}{\partial x_i}\Big)\Big| + \big|\mu_1(\nabla u,\nabla\varphi)\big| \\ &\quad + |\xi(\tau,\nabla\varphi)| + |\langle f,\varphi\rangle|\,dt.\end{aligned}$$

Note that $V \subset L_4$ and $\frac{\xi}{1+\delta(\frac{|\tau|^2}{2\mu_2}+|u|^2)} \leq 1$. Using Hölder's inequality (2.1.1), Cauchy's inequality and estimate (6.3.6), we conclude that the right-hand side does not exceed

$$
\begin{aligned}
&\|\varphi\|_1 \int_0^T \big(\|(u(t)(\cdot), u(t)(\cdot))_{\mathbb{R}^n}\| + \mu_1\|u(t)\|_1 + \|\tau(t)\| + \|f(t)\|_{V^*}\big)dt \\
&\quad \leq \|\varphi\|_1 \int_0^T \big(\|u(t)\|_{L_4}^2 + \frac{1}{2} + \frac{\mu_1}{2}\|u(t)\|_1^2 + \|\tau(t)\| + \|f(t)\|_{V^*}\big)dt \\
&\quad \leq C\|\varphi\|_1\big(1 + \|u\|_{L_2(0,T;V)}^2 + \|\tau\|_{L_\infty(0,T;L_2)} + \|f\|_{L_2(0,T;V^*)}\big) \\
&\quad \leq K_1(\|a\|, \|\tau_0\|, \|f\|_{L_2(0,T;V^*)})\|\varphi\|_1,
\end{aligned}
$$

so we get (6.3.14).

Using embedding $H_0^1 \subset L_4$, (6.3.3) and (6.3.6), we have:

$$
\begin{aligned}
&\|\langle\tau', \Phi\rangle\|_{L_1(0,T)} \\
&\quad = \Big\|\frac{d}{dt}(\tau, \Phi)\Big\|_{L_1(0,T)} \\
&\quad \leq \Big\|\frac{1}{\lambda_1}(\tau, \Phi)\Big\|_{L_1(0,T)} + \frac{\varepsilon}{\lambda_1}\|(\nabla\tau, \nabla\Phi)\|_{L_1(0,T)} \\
&\qquad + \Big\|\xi \sum_{i=1}^n \Big(\frac{u_i\tau}{1+\delta\Big(\frac{|\tau|^2}{2\mu_2}+|u|^2\Big)}, \frac{\partial\Phi}{\partial x_i}\Big)\Big\|_{L_1(0,T)} \\
&\qquad + \|2\xi\mu_2(u, \operatorname{Div}\Phi)\|_{L_1(0,T)} \\
&\quad \leq \frac{1}{\lambda_1}\|\tau\|_{L_1(0,T;L_2)}\|\Phi\|_{L_2} + \frac{\varepsilon}{\lambda_1}\|\Phi\|_1\|\tau\|_{L_1(0,T;H^1)} \\
&\qquad + \|u\|_{L_2(0,T;L_4)}\|\tau\|_{L_2(0,T;L_4)}\Big\|\frac{\partial\Phi}{\partial x_i}\Big\|_{L_2} + 2\mu_2\|u\|_{L_1(0,T;L_2)}\|\operatorname{Div}\Phi\|_{L_2} \\
&\quad \leq C\|\Phi\|_1\big(\|\tau\|_{L_\infty(0,T;L_2)} + \varepsilon\|\tau\|_{L_2(0,T;H^1)} \\
&\qquad + \|u\|_{L_2(0,T;V)}\|\tau\|_{L_2(0,T;V)} + \|u\|_{L_\infty(0,T;L_2)}\big) \\
&\quad \leq \|\Phi\|_1 K_2\big(\|a\|, \|\tau_0\|, \|f\|_{L_2(0,T;V^*)}, \varepsilon\big),
\end{aligned}
$$

and so we arrive at (6.3.15). □

6.4 Passage to the limit.

Consider one more auxiliary system

$$
\frac{d}{dt}(\tau, \Phi) + \frac{1}{\lambda_1}(\tau, \Phi) - \sum_{i=1}^n \big(u_i\tau, \frac{\partial\Phi}{\partial x_i}\big) + 2\mu_2(u, \operatorname{Div}\Phi) + \frac{\varepsilon}{\lambda_1}(\nabla\tau, \nabla\Phi) = 0, \quad (6.4.1)
$$

$$\frac{d}{dt}(u,\varphi) - \sum_{i=1}^{n}\big(u_i u, \frac{\partial \varphi}{\partial x_i}) + \mu_1(\nabla u, \nabla\varphi) + (\tau, \nabla\varphi\big) = \langle f, \varphi\rangle \tag{6.4.2}$$

for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$ almost everywhere in $(0, T)$; $0 < \varepsilon \le 1$.

Theorem 6.4.1. *Let Ω, a, τ_0, f satisfy the conditions of Theorem 6.3.1. Then the problem (6.4.1), (6.4.2), (6.3.5) possesses a solution in the class*

$$\begin{aligned} &u \in L_2(0,T;V), \quad \tau \in L_2(0,T;H_0^1),\\ &\frac{du}{dt} \in L_1(0,T;V^*), \quad \frac{d\tau}{dt} \in L_1(0,T;H^{-1}) \end{aligned} \tag{6.4.3}$$

which satisfies estimates (6.3.14), (6.3.15) and

$$\begin{aligned} &\operatorname{vrai}\max_{t\in[0,T]} \|u\|(t) + \operatorname{vrai}\max_{t\in[0,T]} \|\tau\|(t) + \int_0^T \|u\|_1^2(t)dt + \varepsilon\int_0^T \|\tau\|_1^2(t)dt\\ &\quad \le K_0(\|a\|, \|\tau_0\|, \|f\|_{L_2(0,T;V^*)}). \end{aligned} \tag{6.4.4}$$

Proof. Consider problems (6.3.3) – (6.3.5) with $\xi = 1$ and $\delta = \frac{1}{m}$, $m = 1, 2, \ldots$. By Theorem 6.3.1 there exist solutions (u_m, τ_m) of these problems. Taking into account estimate (6.3.6), without loss of generality (see Remark 2.1.1) we may assume that there exists a pair (u_*, τ_*) such that

$$\begin{aligned} &u_m \to u_* \text{ weakly in } L_2(0,T;V),\\ &u_m \to u_* \text{ *-weakly in } L_\infty(0,T;H),\\ &\tau_m \to \tau_* \text{ weakly in } L_2(0,T;H_0^1),\\ &\tau_m \to \tau_* \text{ *-weakly in } L_\infty(0,T;L_2), \end{aligned}$$

as $m \to \infty$.

By Lemma 6.3.2 the sequence $\left\{\frac{du_m}{dt}\right\}$ is bounded in $L_1(0,T;V^*)$, and the sequence $\left\{\frac{d\tau_m}{dt}\right\}$ is bounded in $L_1(0,T;H^{-1})$. Then by Theorem 2.2.6

$$\begin{aligned} &u_m \to u_* \text{ strongly in } L_2(0,T;H),\\ &\tau_m \to \tau_* \text{ strongly in } L_2(0,T;L_2), \end{aligned}$$

so without loss of generality we may assume that

$$\begin{aligned} &u_m(t)(x) \to u_*(t)(x) \text{ almost everywhere in } (0,T)\times\Omega,\\ &\tau_m(t)(x) \to \tau_*(t)(x) \text{ almost everywhere in } (0,T)\times\Omega. \end{aligned}$$

It is obvious that estimate (6.4.4) is valid for (u_*, τ_*).

Substitute (u_m, τ_m) in equalities (6.3.3) and (6.3.4) with $\delta = \frac{1}{m}$, $\xi = 1$. Taking the scalar product of these equalities in $L_2(0,T)$ with a smooth scalar function $\psi(t)$, $\psi(T) = 0$ and integrating by parts the first terms, we obtain

$$-\int_0^T (\tau_m, \Phi\psi'(t))dt + \int_0^T \Big(\frac{1}{\lambda_1}(\tau_m, \psi\Phi) - \sum_{i=1}^n \Big(\frac{(u_m)_i\ \tau_m}{1+\frac{1}{m}(\frac{|\tau_m|^2}{2\mu_2} + |u_m|^2)}, \psi\frac{\partial\Phi}{\partial x_i}\Big) + 2\mu_2(u_m, \psi\operatorname{Div}\Phi) + \frac{\varepsilon}{\lambda_1}(\nabla\tau_m, \psi\nabla\Phi)\Big)dt = (\tau_0, \Phi)\psi(0), \tag{6.4.5}$$

$$-\int_0^T (u_m, \varphi\psi'(t))dt + \int_0^T \Big(\mu_1(\nabla u_m,\ \psi\nabla\varphi) - \sum_{i=1}^n \Big(\frac{(u_m)_i\ u_m}{1+\frac{1}{m}(\frac{|\tau_m|^2}{2\mu_2} + |u_m|^2)}, \psi\frac{\partial\varphi}{\partial x_i}\Big) + (\tau_m, \psi\nabla\varphi)\Big)\, dt = \int_0^T \langle f, \varphi\psi\rangle dt + (a,\varphi)\psi(0). \tag{6.4.6}$$

Let us check that

$$\int_0^T \sum_{i=1}^n \Big(\frac{(u_m)_i\ \tau_m}{1+\frac{1}{m}(\frac{|\tau_m|^2}{2\mu_2} + |u_m|^2)}, \psi\frac{\partial\Phi}{\partial x_i}\Big)dt \to \int_0^T \sum_{i=1}^n \Big((u_*)_i\tau_*,\ \psi\frac{\partial\Phi}{\partial x_i}\Big)dt, \tag{6.4.7}$$

$$\int_0^T \sum_{i=1}^n \Big(\frac{(u_m)_i u_m}{1+\frac{1}{m}(\frac{|\tau_m|^2}{2\mu_2} + |u_m|^2)}, \psi\frac{\partial\varphi}{\partial x_i}\Big)dt \to \int_0^T \sum_{i=1}^n \Big((u_*)_i u_*, \psi\frac{\partial\varphi}{\partial x_i}\Big)dt \tag{6.4.8}$$

as $m \to \infty$.

We have:

$$\Big|\int_0^T \sum_{i=1}^n \Big(\frac{(u_m)_i\ \tau_m}{1+\frac{1}{m}(\frac{|\tau_m|^2}{2\mu_2} + |u_m|^2)} - (u_*)_i\tau_*, \psi\frac{\partial\Phi}{\partial x_i}\Big)dt\Big|$$

$$= \Big|\int_0^T \sum_{i,j,k=1}^n \Big(\frac{(u_m)_i\ (\tau_m)_{jk}}{1+\frac{1}{m}(\frac{|\tau_m|^2}{2\mu_2} + |u_m|^2)} - (u_*)_i(\tau_*)_{jk}, \psi\frac{\partial\Phi_{jk}}{\partial x_i}\Big)dt\Big|$$

$$\leq C \max_{i,j,k} \Big\| \frac{(u_m)_i\ (\tau_m)_{jk}}{1+\frac{1}{m}\big(\frac{|\tau_m|^2}{2\mu_2}+|u_m|^2\big)} - (u_*)_i (\tau_*)_{jk} \Big\|_{L_1(0,T;L_1)}$$

$$\cdot \|\psi\|_{L_\infty(0,T;\mathbb{R})} \|\Phi\|_{C^1(\Omega,\mathbb{R}_S^{n\times n})}.$$

Then,

$$\Big\| \frac{(u_m)_i\ (\tau_m)_{jk}}{1+\frac{1}{m}\big(\frac{|\tau_m|^2}{2\mu_2}+|u_m|^2\big)} - (u_*)_i (\tau_*)_{jk} \Big\|_{L_1(0,T;L_1)}$$

$$\leq \Big\| \frac{(u_m)_i\ (\tau_m)_{jk} - (u_*)_i (\tau_*)_{jk}}{1+\frac{1}{m}\big(\frac{|\tau_m|^2}{2\mu_2}+|u_m|^2\big)} \Big\|_{L_1(0,T;L_1)}$$

$$+ \Big\| \frac{(u_*)_i (\tau_*)_{jk}}{1+\frac{1}{m}\big(\frac{|\tau_m|^2}{2\mu_2}+|u_m|^2\big)} - (u_*)_i (\tau_*)_{jk} \Big\|_{L_1(0,T;L_1)}$$

$$\leq \big\|(u_m)_i\ (\tau_m)_{jk} - (u_*)_i (\tau_*)_{jk}\big\|_{L_1(0,T;L_1)}$$

$$+ \Big\| \frac{\big(\frac{|\tau_m|^2}{2\mu_2}+|u_m|^2\big)(u_*)_i (\tau_*)_{jk}}{m+\frac{|\tau_m|^2}{2\mu_2}+|u_m|^2} \Big\|_{L_1(0,T;L_1)}$$

$$\leq \|(u_m)_i\|_{L_2(0,T;L_2)} \|(\tau_m)_{jk} - (\tau_*)_{jk}\|_{L_2(0,T;L_2)}$$

$$+ \|(u_m)_i - (u_*)_i\|_{L_2(0,T;L_2)} \|(\tau_*)_{jk}\|_{L_2(0,T;L_2)}$$

$$+ \Big\| \frac{\big(\frac{|\tau_m|^2}{2\mu_2}+|u_m|^2\big)(u_*)_i (\tau_*)_{jk}}{m+\frac{|\tau_m|^2}{2\mu_2}+|u_m|^2} \Big\|_{L_1(0,T;L_1)}.$$

The first and the second terms tend to zero as $m \to \infty$ for $u_m \to u_*$ in $L_2(0,T;H)$, and $\tau_m \to \tau_*$ in $L_2(0,T;L_2)$. It remains to prove that the third term also tends to zero. In fact, the convergences $u_m(t)(x) \to u_*(t)(x)$ almost everywhere in $(0,T)\times\Omega$, $\tau_m(t)(x) \to \tau_*(t)(x)$ almost everywhere in $(0,T)\times\Omega$ imply that

$$\frac{\big(\frac{|\tau_m(t)(x)|^2}{2\mu_2}+|u_m(t)(x)|^2\big)(u_*)_i(t)(x)(\tau_*)_{jk}(t)(x)}{m+\frac{|\tau_m(t)(x)|^2}{2\mu_2}+|u_m(t)(x)|^2}$$

$$\to 0 \text{ almost everywhere in } (0,T)\times\Omega.$$

Furthermore,

$$\Big|\frac{\big(\frac{|\tau_m(t)(x)|^2}{2\mu_2}+|u_m(t)(x)|^2\big)(u_*)_i(t)(x)(\tau_*)_{jk}(t)(x)}{m+\frac{|\tau_m(t)(x)|^2}{2\mu_2}+|u_m(t)(x)|^2}\Big| \leq \big|(u_*)_i(t)(x)(\tau_*)_{jk}(t)(x)\big|,$$

and $(u_*)_i(\tau_*)_{jk}(t)(x) \in L_1((0,T)\times\Omega)$. Thus, by the Lebesgue theorem,

$$\frac{\left(\frac{|\tau_m(t)(x)|^2}{2\mu_2} + |u_m(t)(x)|^2\right)(u_*)_i(t)(x)(\tau_*)_{jk}(t)(x)}{m + \frac{|\tau_m(t)(x)|^2}{2\mu_2} + |u_m(t)(x)|^2} \to 0$$

in $L_1((0,T)\times\Omega)$ as $m\to\infty$.

We have proved (6.4.7). Observe that (6.4.8) can be shown just in the same way.

Now, passing to the limit in (6.4.5), (6.4.6) as $m\to\infty$, we conclude

$$\begin{aligned} &-\int_0^T (\tau_*, \Phi\psi'(t))dt + \int_0^T \Big(\frac{1}{\lambda_1}(\tau_*, \psi\Phi) - \sum_{i=1}^n \Big((u_*)_i\tau_*, \psi\frac{\partial\Phi}{\partial x_i}\Big) \\ &\quad + 2\mu_2(u_*, \psi\operatorname{Div}\Phi) + +\frac{\varepsilon}{\lambda_1}(\nabla\tau_*, \psi\nabla\Phi)\Big)dt = (\tau_0,\Phi)\psi(0), \end{aligned} \tag{6.4.9}$$

$$\begin{aligned} &-\int_0^T (u_*, \varphi\psi'(t))dt + \int_0^T \Big(\mu_1(\nabla u_*, \psi\nabla\varphi) - \sum_{i=1}^n \Big((u_*)_i u_*, \psi\frac{\partial\varphi}{\partial x_i}\Big) \\ &\quad + (\tau_*, \psi\nabla\varphi)\Big)dt = \int_0^T \langle f, \varphi\psi\rangle dt + (a,\varphi)\psi(0). \end{aligned} \tag{6.4.10}$$

Since it has been carried out, in particular, for every $\psi \in C_0^\infty(0,T)$, the function (u_*,τ_*) satisfies (6.4.1), (6.4.2) in the sense of distributions on $(0,T)$.

Substitute (u_*,τ_*) into equalities (6.4.1), (6.4.2). Since all terms in the obtained equalities are integrable on $(0,T)$, these equalities are valid almost everywhere on $(0,T)$. Taking the scalar product of these equalities in $L_2(0,T)$ with a smooth scalar function $\psi(t),\ \psi(T)=0,\ \psi(0)\neq 0$ and comparing the result with (6.4.9), (6.4.10), we see that

$$(u_*|_{t=0},\varphi)\psi(0) = (a,\varphi)\psi(0),$$
$$(\tau_*|_{t=0},\ \Phi)\psi(0) = (\tau_0,\Phi)\psi(0).$$

Since Φ and φ are arbitrary, u_* and τ_* satisfy (6.3.5). Repeating the proof of Lemma 6.3.2 with $\delta=0,\ \xi=1$, we see that the solutions of problem (6.4.1), (6.4.2), (6.3.5) satisfy estimates (6.3.14), (6.3.15). Thus, (u_*,τ_*) is the desirable solution. □

Remark 6.4.1. There is another way to check that the limit (u_*,τ_*) satisfies the initial condition (6.3.5). One has to observe that the the sequence $\{\frac{du_m}{dt}\}$ is in fact bounded in $L_{4/3}(0,T;V^*)$, and the sequence $\{\frac{d\tau_m}{dt}\}$ is bounded in $L_2(0,T;H^{-2})$ (see Remark 6.8.1 below). But since $u_m \to u_*$ *-weakly in $L_\infty(0,T;H)$, $\tau_m\to\tau_*$ *-weakly in $L_\infty(0,T;L_2)$, Theorem 2.2.6 gives that $u_m\to u_*$ strongly in $C(0,T;V^*)$,

$\tau_m \to \tau_*$ strongly in $C(0,T;H^{-1})$. In particular, there is the pointwise convergence. Hence, $a = u_m(0) \to u_*(0)$ strongly in V^*, i.e. $a = u_*(0)$, and $\tau_0 = \tau_m(0) \to \tau_*(0)$ strongly in H^{-1}, i.e. $\tau_0 = \tau_*(0)$. Such an approach may be used also in the proof of Theorem 6.5.1, where we, however, shall apply the first way of proof again.

6.5 Existence of a weak solution for the Jeffreys model

6.5.1 Existence of velocity and stress

Let us prove a statement which yields Theorem 6.2.1 immediately.

Theorem 6.5.1 (see [72]). *Given $f \in L_2(0,T;V^*)$, $a \in H$, $\tau_0 \in L_2$, there exists a pair of functions (u,τ),*

$$\begin{aligned} &u \in L_2(0,T;V) \bigcap L_\infty(0,T;H) \bigcap C_w([0,T],H),\ \frac{du}{dt} \in L_1(0,T;V^*),\\ &\tau \in L_\infty(0,T;L_2) \bigcap C_w([0,T],L_2),\ \frac{d\tau}{dt} \in L_2(0,T;H^{-2}) \end{aligned} \tag{6.5.1}$$

satisfying (6.3.1), (6.3.2) almost everywhere in $(0,T)$, the initial condition (6.3.5) and the estimate

$$\begin{aligned} &\|u\|_{L_2(0,T;V)} + \|u\|_{L_\infty(0,T;H)} + \Big\|\frac{du}{dt}\Big\|_{L_1(0,T;V^*)} + \|\tau\|_{L_\infty(0,T;L_2)}\\ &\quad + \Big\|\frac{d\tau}{dt}\Big\|_{L_2(0,T;H^{-2})} \le K_3(\|a\|, \|\tau_0\|, \|f\|_{L_2(0,T;V^*)}), \end{aligned} \tag{6.5.2}$$

where K_3 does not depend on Ω.

Remark 6.5.1. As a matter of fact, the solution (u,τ) satisfies (6.3.1) for all $\Phi \in \overset{\circ}{W}{}^1_4(\Omega, \mathbb{R}^{n\times n}_S)$ (not only from C_0^∞), and (6.3.2) holds for all $\varphi \in V$ (not only from $\mathcal{V}$). Really, as $u \in L_2(0,T;V) \subset L_2(0,T;H_0^1) \subset L_2(0,T;L_4)$, one has $u_i u \in L_1(0,T;L_2)$ by inequality (2.2.1). But $\nabla u \in L_2(0,T;L_2)$, $\tau \in L_\infty(0,T;L_2)$, $f \in L_2(0,T;V^*)$. Applying Lemma 2.2.8 (implication i) $\to$ ii)) with $X = V^*$, $X^* = V$ (V is reflexive), $Z = \mathcal{V}$, and with (6.3.2) instead of (2.2.29), we get the second claim. Furthermore, since $u \in L_2(0,T;L_4)$, $\tau \in L_\infty(0,T;L_2)$, one has $u_i\tau \in L_2(0,T;L_{4/3})$; and applying Lemma 2.2.8 with $X = W^{-1}_{4/3}(\Omega, \mathbb{R}^{n\times n}_S)$, $X^* = \overset{\circ}{W}{}^1_4(\Omega, \mathbb{R}^{n\times n}_S)$, $Z = C_0^\infty$, and with (6.3.1) instead of (2.2.29), we get the first claim.

Proof of Theorem 6.5.1. Denote by Ω_m the intersection of Ω with the ball B_m centered at the origin in the space $\mathbb{R}^n$ of radius $m = 1, 2, \ldots$. From the definition of the space H (Section 2.1.2) it easily follows that there exists a sequence $\{a_m\}$,

$$a_m \in C_0^\infty(\Omega)^n,\ \operatorname{div} a_m = 0,\ \operatorname{supp} a_m \subset \Omega_m,\ a_m \to a \text{ in } L_2(\Omega)^n,\ \|a_m\| \le \|a\|.$$

Consider on Ω_m, for every m, problem (6.4.1), (6.4.2) with $\varepsilon = \frac{1}{m}$ and the initial condition

$$u|_{t=0} = a_m, \ \tau|_{t=0} = \tau_0|_{\Omega_m}. \tag{6.5.3}$$

By Theorem 6.4.1 there exists a solution (u_m, τ_m) of this problem. All these solutions are bounded by estimates (6.3.14) and (6.4.4) with $\varepsilon = \frac{1}{m}$. Denote by $\tilde{u}_m$ and $\tilde{\tau}_m$ the functions which coincide with u_m and τ_m, respectively, in Ω_m, and are identically zero in $\Omega \backslash \Omega_m$. Without loss of generality (see Remark 2.1.1) we may assume that there exists a pair (u_*, τ_*) such that

$$\begin{aligned}
\tilde{u}_m &\to u_* \text{ weakly in } L_2(0,T;V),\\
\tilde{u}_m &\to u_* \ \text{*-weakly in } \ L_\infty(0,T;H),\\
\tilde{\tau}_m &\to \tau_* \ \text{*-weakly in } L_\infty(0,T;L_2).
\end{aligned}$$

Furthermore, by Theorem 2.2.6:

$$\tilde{u}_m|_{\Omega_k} \to u_*|_{\Omega_k} \text{ strongly in } L_2(0,T;L_2(\Omega_k))$$

for every k. Obviously, (u_*, τ_*) satisfies estimate (6.4.4).

Take arbitrary $\varphi \in \mathcal{V}$, $\Phi \in C_0^\infty$. Fix k large enough, such that the supports of φ and Φ are contained in Ω_k.

Substitute (u_m, τ_m) in equalities (6.4.1), (6.4.2) with $m \geq k$, $\varepsilon = \frac{1}{m}$. Take the scalar product of these equalities in $L_2(0,T)$ with a smooth scalar function $\psi(t)$, $\psi(T) = 0$ and integrate by parts the first terms. Due to the choice of k we can replace u_m and τ_m in these equalities by $\tilde{u}_m$ and $\tilde{\tau}_m$. We have:

$$\begin{aligned}
-\int_0^T (\tilde{\tau}_m,\ \Phi\varphi'(t))dt &+ \int_0^T \Big(\frac{1}{\lambda_1}(\tilde{\tau}_m, \psi\Phi) - \sum_{i=1}^n \Big((\tilde{u}_m)_i\ \tilde{\tau}_m,\ \psi\frac{\partial\Phi}{\partial x_i}\Big)\\
&+ 2\mu_2(\tilde{u}_m, \psi \operatorname{Div}\Phi) + \frac{1}{\lambda_1 m}(\nabla\tilde{\tau}_m, \psi\nabla\Phi)\Big)dt = (\tau_0, \Phi)\psi(0),
\end{aligned} \tag{6.5.4}$$

$$\begin{aligned}
-\int_0^T \Big(\tilde{u}_m, \varphi\psi'(t)\Big)dt &+ \int_0^T \Big(\mu_1(\nabla\tilde{u}_m, \psi\nabla\varphi) - \sum_{i=1}^n \Big((\tilde{u}_m)_i\tilde{u}_m, \psi\frac{\partial\varphi}{\partial x_i}\Big)\\
&+ (\tilde{\tau}_m, \psi\nabla\varphi)\Big)\,dt = \int_0^T \langle f, \varphi\psi\rangle dt + (a_m, \varphi)\psi(0).
\end{aligned} \tag{6.5.5}$$

Observe that

$$\begin{aligned}
\Big|\frac{1}{m}\int_0^T (\nabla\tilde{\tau}_m, \psi\nabla\Phi)\Big| &= \Big|\frac{1}{m}\int_0^T (\tilde{\tau}_m, \psi\Delta\Phi)\Big|\\
&\leq \frac{1}{m}\|\tau_m\|_{L_\infty(0,T;L_2)}\int_0^T \|\psi\Delta\Phi\| \underset{m\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Now, let us show that

$$\int_0^T \sum_{i=1}^n \Big((\tilde{u}_m)_i \tilde{\tau}_m, \psi \frac{\partial \Phi}{\partial x_i}\Big) dt \to \int_0^T \sum_{i=1}^n \Big((\tilde{u}_*)_i \tilde{\tau}_*, \psi \frac{\partial \Phi}{\partial x_i}\Big) dt, \tag{6.5.6}$$

$$\int_0^T \sum_{i=1}^n \Big((\tilde{u}_m)_i \tilde{u}_m, \psi \frac{\partial \varphi}{\partial x_i}\Big) dt \to \int_0^T \sum_{i=1}^n \Big((\tilde{u}_*)_i \tilde{u}_*, \psi \frac{\partial \varphi}{\partial x_i}\Big) dt. \tag{6.5.7}$$

Really, using inequality (2.2.1), we get:

$$\begin{aligned}
&\Big|\int_0^T \sum_{i=1}^n \Big((\tilde{u}_m)_i \tilde{\tau}_m - (\tilde{u}_*)_i \tilde{\tau}_*, \psi \frac{\partial \Phi}{\partial x_i}\Big) dt\Big| \\
&\le \Big|\int_0^T \sum_{i=1}^n \Big(((\tilde{u}_m)_i - (\tilde{u}_*)_i)\tilde{\tau}_m, \psi \frac{\partial \Phi}{\partial x_i}\Big) dt\Big| \\
&\qquad + \Big|\int_0^T \sum_{i=1}^n \Big((\tilde{u}_*)_i (\tilde{\tau}_m - \tilde{\tau}_*), \psi \frac{\partial \Phi}{\partial x_i}\Big) dt\Big| \\
&\le \|(u_m - u_*)\Big|_{\Omega_k}\|_{L_2(0,T;L_2)} \|\tau_m\|_{L_2(0,T;L_2)} \\
&\qquad \times \|\psi \nabla \Phi\|_{L_\infty(0,T;L_\infty)} + \Big|\Big\langle \tilde{\tau}_m - \tilde{\tau}_*, \sum_{i=1}^n (\tilde{u_*})_i \psi \frac{\partial \Phi}{\partial x_i}\Big\rangle_{L_\infty(0,T;L_2)\times L_1(0,T;L_2)}\Big|.
\end{aligned}$$

Both terms tend to zero, and (6.5.6) is proved. Similarly one shows (6.5.7).

Now, let m tend to infinity in (6.5.4) and (6.5.5). We obtain:

$$\begin{aligned}
&-\int_0^T (\tau_*, \Phi \psi'(t)) dt + \int_0^T \Big(\frac{1}{\lambda_1}(\tau_*, \psi \Phi) - \sum_{i=1}^n \Big((u_*)_i \tau_*, \psi \frac{\partial \Phi}{\partial x_i}\Big) \\
&\qquad + 2\mu_2(u_*, \psi \operatorname{Div} \Phi)\Big) dt = (\tau_0, \Phi)\psi(0),
\end{aligned} \tag{6.5.8}$$

$$\begin{aligned}
&-\int_0^T \Big(u_*, \varphi \psi'(t)\Big) dt + \int_0^T \Big(\mu_1(\nabla u_*, \psi \nabla \varphi) - \sum_{i=1}^n \Big((u_*)_i u_*, \psi \frac{\partial \varphi}{\partial x_i}\Big) \\
&\qquad + (\tau_*, \psi \nabla \varphi)\Big) dt = \int_0^T \langle f, \varphi \psi \rangle dt + (a, \varphi)\psi(0).
\end{aligned} \tag{6.5.9}$$

As in the proof of Theorem 6.4.1, this implies that (u_*, τ_*) is a solution of (6.3.1), (6.3.2), (6.3.5) and u_* satisfies estimate (6.3.14).

To prove (6.5.2), it remains to estimate the fifth term in its left-hand side. Substitute (u_*, τ_*) in (6.3.1). Using inequality (2.2.1), the embedding $V \subset L_4$, and estimate

(6.4.4) for (u_*, τ_*) , we have:

$$
\begin{aligned}
\|\langle\tau_*', \Phi\rangle\|_{L_2(0,T)} &= \Big\|\frac{d}{dt}(\tau_*, \Phi)\Big\|_{L_2(0,T)} \\
&\leq \Big\|\frac{1}{\lambda_1}(\tau_*, \Phi)\Big\|_{L_2(0,T)} + \Big\|\sum_{i=1}^{n}((u_*)_i\tau_*, \frac{\partial\Phi}{\partial x_i})\Big\|_{L_2(0,T)} \\
&\quad + \|2\mu_2(u_*, \operatorname{Div}\Phi)\|_{L_2(0,T)} \\
&\leq \frac{1}{\lambda_1}\|\tau_*\|_{L_2(0,T.L_2)}\|\Phi\|_{L_2} + \|u_*\|_{L_2(0,T;L_4)}\|\tau_*\|_{L_\infty(0,T;L_2)}\big\|\nabla\Phi\big\|_{L_4} \\
&\quad + 2\mu_2\|u_*\|_{L_2(0,T;L_2)}\|\operatorname{Div}\Phi\|_{L_2} \\
&\leq C\|\Phi\|_{H_0^2}(\|\tau_*\|_{L_\infty(0,T;L_2)} + \|u_*\|_{L_2(0,T;V)}\|\tau_*\|_{L_\infty(0,T;L_2)} \\
&\quad + \|u_*\|_{L_2(0,T;V)}) \\
&\leq K_4(\|a\|,\ \|\tau_0\|,\ \|f\|_{L_2(0,T;V^*)})\|\Phi\|_{H_0^2},
\end{aligned}
$$

and we obtain the desirable estimate.

By Lemma 2.2.8, almost everywhere on $(0, T)$ u_* is equal to a continuous function with values in V^*. Thus, without loss of generality we may assume that u_* itself is continuous on $[0, T]$ with values in this space. And since it belongs to $L_\infty(0, T; H)$, by Lemma 2.2.6 it is weakly continuous on $[0, T]$ as a function with values in H. Similarly, $\tau \in C_w([0, T]; L_2)$. The proof of the theorem is complete. □

Remark 6.5.2. By the same scheme it is easy to show that in Theorems 6.3.1 and 6.4.1 the condition of boundedness of Ω is not necessary.

Proof of Theorem 6.2.1. Take $\tau_0 = \sigma_0 - 2\mu_1\mathcal{E}(a)$. By the condition of the theorem $\tau_0 \in L_2$. By Theorem 6.5.1 there exists a solution (u, τ) of problem (6.3.1), (6.3.2), (6.3.5) in class (6.5.1).

It is easy to see that $\mathcal{E}(u) \in L_2(0, T; L_2) \bigcap C_w([0, T]; H^{-1})$.

Take $\sigma = \tau + 2\mu_1\mathcal{E}(u)$. Then the pair (u, σ) belongs to class (6.2.6) and satisfies (6.2.5), (6.2.7), (6.2.8), i.e. it is a weak solution of problem (6.2.1) – (6.2.5). □

Remark 6.5.3. Since (6.3.1) and (6.3.2) are satisfied almost everywhere in $(0, T)$, (6.2.7) is also satisfied almost everywhere in $(0, T)$. But we cannot assert that (6.2.8) is also satisfied almost everywhere in $(0, T)$, since (6.2.8) contains two time derivatives, and Lemma 2.2.8 is not applicable here.

6.5.2 Existence of pressure

The pressure p was removed from problem (6.2.1) – (6.2.5) by the weak setting procedure. Condition (6.1.4) gives an opportunity to restore it when a weak solution is sufficiently regular. Is it possible to restore the pressure (maybe in some generalized sense) in general? In this subsection we examine this question.

We begin with a "positive" result.

Theorem 6.5.2. *Let Ω be as in Corollary 3.1.1, $n = 2, 3$, and let $f \in L_2(0, T; H^{-1}(\Omega)^n)$. Then for any weak solution (u, σ) to problem (6.2.1) – (6.2.5) there exists a function $p \in W_\infty^{-1}(0, T; L_2(\Omega))$ such that equality (6.2.1) holds (e.g. in the space $W_\infty^{-1}(0, T; H^{-1}(\Omega)^n)$).*

Remark 6.5.4. Under the conditions of the theorem, use of Definition 6.2.1 is possible since one can always assume that $f \in L_2(0, T; V^*)$. Really, for almost all $t \in (0, T)$,

$$f(t) = (f_1(t), \ldots, f_n(t)) \in H^{-1}(\Omega)^n = ((H_0^1(\Omega))^*)^n.$$

Then $f(t)$ may be considered as a linear continuous functional on V (i.e. as an element of V^*) according to the formula

$$\langle f(t), \varphi \rangle_{V^* \times V} = \sum_{i=1}^{n} \langle f_i(t), \varphi_i \rangle_{H^{-1} \times H_0^1}, \ \varphi \in V.$$

Proof of Theorem 6.5.2. Let (u, σ) be a weak solution to problem (6.2.1) – (6.2.5). Then, in particular,

$$u \in L_2(0, T; V) \bigcap L_\infty(0, T; H),$$
$$\sigma \in L_2(0, T; L_2(\Omega, \mathbb{R}_S^{n \times n})),$$

and

$$\frac{d}{dt}(u, \varphi) + (\sigma, \nabla\varphi) - \sum_{i=1}^{n} \big(u_i u, \frac{\partial \varphi}{\partial x_i}\big) = \langle f, \varphi \rangle \tag{6.5.10}$$

for all $\varphi \in \mathcal{V}$.

Thus,

$$u \in L_2(0, T; L_4), \quad \nabla u \in L_2(0, T; L_2).$$

Then by (2.2.1),

$$u_i \frac{\partial u}{\partial x_i} \in L_1(0, T; L_{4/3}) \subset L_1(0, T; H^{-1}) \subset W_\infty^{-1}(0, T; H^{-1}), \quad i = 1, \ldots, n.$$

The first embedding follows from Theorem 2.1.1 a). Let us check the second one. By Corollary 2.2.1,

$$W_1^1(0, T; H_0^1) \subset C([0, T]; H_0^1).$$

The embedding is dense and continuous. Hence,

$$L_1(0,T;H^{-1}) \subset (C([0,T];H_0^1))^* \subset (W_1^1(0,T;H_0^1))^* = W_\infty^{-1}(0,T;H^{-1}).$$

Observe that

$$u' \in W_\infty^{-1}(0,T;L_2),$$
$$\text{Div}\ \sigma \in L_2(0,T;H^{-1}).$$

Integrating by parts (see Section 6.1.2) in the second and the third terms of (6.5.10) we conclude that

$$(u',\varphi) - (\text{Div}\ \sigma,\varphi) + \Big(\sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i},\varphi\Big) = \langle f,\varphi\rangle \qquad (6.5.11)$$

for all $\varphi \in \mathcal{V}$. Let $g = f - u' + \text{Div}\ \sigma - \sum\limits_{i=1}^{n} u_i \frac{\partial u}{\partial x_i}$. Then $g \in W_\infty^{-1}(0,T;H^{-1}(\Omega)^n)$, and

$$\langle g(t),\varphi\rangle = 0 \qquad (6.5.12)$$

for all $\varphi \in \mathcal{V}$. By Corollary 3.1.3, there exists $p \in W_\infty^{-1}(0,T;L_2)$ such that

$$\text{grad}\ p = g,$$

which is equivalent to (6.2.1). The proof is complete. □

However the answer to the question which was put at the beginning of this subsection is not positive in general. Even in the case of bounded sufficiently regular domain Ω with connected boundary, (6.2.1) cannot hold a priori for body forces from $L_2(0,T;V^*(\Omega))$. Let us illustrate this using the ideas from [55].

Lemma 6.5.1 (see [55], p. 228). *If Ω is bounded sufficiently regular and its boundary is connected, there exists a sequence of functions $\{v_m\} \subset \mathcal{V}$ such that*

$$v_m \to 0 \ \ in \ \ V^*, \quad v_m \to v \neq 0 \ \ in \ \ H^{-1}(\Omega)^n. \qquad (6.5.13)$$

Corollary 6.5.1. *Let Ω be bounded sufficiently regular and let its boundary be connected.*

a) *There is no Hausdorff topological vector space $\mathcal{Y}$ such that*

$$V^*(\Omega) \subset \mathcal{Y}, \quad H^{-1}(\Omega)^n \subset \mathcal{Y}$$

with continuous embeddings;

b) *there is no Hausdorff topological vector space* $\mathfrak{Y}$ *such that*

$$L_2(0,T;V^*(\Omega)) \subset \mathfrak{Y}, \quad L_2(0,T;H^{-1}(\Omega)^n) \subset \mathfrak{Y}$$

with continuous embeddings.

Since different members of (6.2.1) belong to different spaces (in particular, $u' \in W_\infty^{-1}(0,T;L_2)$, $\operatorname{Div}\sigma \in L_2(0,T;H^{-1}(\Omega)^n)$, $f \in L_2(0,T;V^*)$), one needs some "large" uniting space which contains them all (in order to interpret equality (6.2.1) in this space). Due to Corollary 6.5.1 b), it turns out to be impossible to find such a space even with minimal restrictions of its properties.

However, a "uniting" space exists for $\Omega = \mathbb{R}^n$: for example, it is the space $L_1(0,T;H^{-1}(\mathbb{R}^n)^n)$. Observe first that for $u \in H(\mathbb{R}^n)$, $\varphi \in H^1(\mathbb{R}^n,\mathbb{R}^n)$ one has

$$\langle u,\varphi\rangle_{H^{-1}\times H^1} = (u,\varphi) = (u,P\varphi) = \langle u,P\varphi\rangle_{H^*\times H} = \langle u,P\varphi\rangle_{V^*\times V},$$

where P is the Leray projection (5.2.12). Note that Lemma 3.1.1 and Liouville's theorem imply

$$P(L_2(\mathbb{R}^n)) = H_V^0 = H(\mathbb{R}^n),\ P(H^1(\mathbb{R}^n)) = H_V^1 = V(\mathbb{R}^n).$$

Thus, we arrive at the embedding

$$V^*(\mathbb{R}^n) \subset H^{-1}(\mathbb{R}^n)^n, \tag{6.5.14}$$

i.e. any $u \in V^*$ determines a linear continuous functional on $H^1(\mathbb{R}^n)^n$ by the formula

$$\langle u,\varphi\rangle_{H^{-1}(\mathbb{R}^n)^n\times H^1(\mathbb{R}^n)^n} = \langle u,P\varphi\rangle_{V^*\times V} \tag{6.5.15}$$

for $\varphi \in H^1(\mathbb{R}^n)^n$.

Theorem 6.5.3. *Let $f \in L_2(0,T;V^*(\mathbb{R}^n)^n)$. Then for any weak solution (u,σ) to problem (6.2.1) – (6.2.5) there exists a function $p \in L_1(0,T;L_{2,\mathrm{loc}}(\mathbb{R}^n))$ such that equality (6.2.1) holds (e.g. in the space $L_1(0,T;H^{-1}(\mathbb{R}^n)^n)$).*

Proof. Let (u,σ) be a weak solution to problem (6.2.1) – (6.2.5). Then, due to (6.5.14),

$$u' \in L_1(0,T;V^*) \subset L_1(0,T;H^{-1}),$$
$$f \in L_2(0,T;V^*) \subset L_2(0,T;H^{-1}).$$

But as in the proof of Theorem 6.5.2,

$$u_i\frac{\partial u}{\partial x_i} \in L_1(0,T;H^{-1}),$$

$$\operatorname{Div}\sigma \in L_2(0,T;H^{-1}),$$

and one has (6.5.10) and (6.5.11).

Let $g = f - u' + \mathrm{Div}\,\sigma - \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i}$. Then $g \in L_1(0,T;H^{-1}(\mathbb{R}^n)^n)$, and (6.5.12) holds for all $\varphi \in \mathcal{V}$. By Corollary 3.1.4, there exists $p \in L_1(0,T;L_{2,\mathrm{loc}})$ such that

$$\mathrm{grad}\; p = g,$$

which is equivalent to (6.2.1). □

6.6 Differential energy estimate and uniqueness of the weak solution for the Jeffreys model

The problem of uniqueness of weak solutions for the majority of the equations of hydrodynamics generally remains open. For example, for the equations of Navier–Stokes (Newtonian fluid) in the two-dimensional case a weak solution is unique, and in the three dimensions there are only conditional results. For instance, the classical result by Sather and Serrin ([61], Theorem III.3.9) says that if a weak solution to the initial-boundary value for the equations of Navier–Stokes belongs, in addition, to $L_8(0,T;L_4)$, then it is unique in the class of weak solutions satisfying an energy inequality. In this section we shall prove a similar result for the Jeffreys model.

6.6.1 Differential energy inequality

First we derive an energy inequality for the constructed in Theorem 6.4.1 solution to problem (6.4.1), (6.4.2), (6.3.5).

Lemma 6.6.1. *The solution to problem (6.4.1), (6.4.2), (6.3.5) constructed in Theorem 6.4.1 satisfies the following inequality for almost all $t \in [0,T]$:*

$$\begin{aligned}\frac{1}{2}\|u_*\|^2(t) + \frac{1}{4\mu_2}\|\tau_*\|^2(t) + \int_0^t \frac{1}{2\lambda_1\mu_2}\big((1-\varepsilon)\|\tau_*\|^2 + \varepsilon\|\tau_*\|_1^2\big)\,ds \\ + \int_0^t \mu_1\|u_*\|_Y^2\,ds \le \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \int_0^t \langle f(s), u_*(s)\rangle\,ds.\end{aligned} \tag{6.6.1}$$

Remark 6.6.1. The space $Y = Y(\Omega)$ was introduced in Section 2.1.2 and will be often used from now on.

Proof. Consider the pairs (u_m, τ_m) which were used in the proof of Theorem 6.4.1 for the construction of the solution. Each of them satisfies equality (6.3.8). Having integrated this equality from 0 to t, we get:

$$\frac{1}{2}\|u_m\|^2(t) + \frac{1}{4\mu_2}\|\tau_m\|^2(t) + \int_0^t \frac{1}{2\lambda_1\mu_2}\left((1-\varepsilon)\|\tau_m\|^2 + \varepsilon\|\tau_m\|_1^2\right) ds$$
$$+ \int_0^t \mu_1\|u_m\|_Y^2\, ds = \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \int_0^t \langle f(s), u_m(s)\rangle\, ds.$$

Take the scalar product in $L_2(0,T)$ of this equality with a smooth function $\psi \in \mathcal{D}(0,T)$ with non-negative values:

$$\int_0^T \Big\{\frac{1}{2}\|u_m\|^2(t) + \frac{1}{4\mu_2}\|\tau_m\|^2(t) + \int_0^t \frac{1}{2\lambda_1\mu_2}\left((1-\varepsilon)\|\tau_m\|^2 + \varepsilon\|\tau_m\|_1^2\right) ds$$
$$+ \int_0^t \mu_1\|u_m\|_Y^2\, ds\Big\}\psi(t)\, dt$$
$$= \int_0^T \Big(\frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \int_0^t \langle f(s), u_m(s)\rangle\, ds\Big)\psi(t)\, dt.$$

Passing in this relation to the inferior limit as $m \to \infty$, and using the fact that the norm of a weak limit of a sequence does not exceed the inferior limit of the norms, we arrive at:

$$\int_0^T \Big\{\frac{1}{2}\|u_*\|^2(t) + \frac{1}{4\mu_2}\|\tau_*\|^2(t) + \int_0^t \frac{1}{2\lambda_1\mu_2}\left((1-\varepsilon)\|\tau_*\|^2 + \varepsilon\|\tau_*\|_1^2\right) ds$$
$$+ \int_0^t \mu_1\|u_*\|_Y^2\, ds\Big\}\psi(t)\, dt$$
$$\le \int_0^T \Big(\frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \int_0^t \langle f(s), u_*(s)\rangle\, ds\Big)\psi(t)\, dt.$$

Since ψ was chosen arbitrarily, this yields (6.6.1). □

It is also possible to get an energy inequality for the solution of problem (6.3.1), (6.3.2), (6.3.5):

Lemma 6.6.2. *The solution to problem (6.3.1), (6.3.2), (6.3.5) constructed in Theorem 6.5.1 satisfies the following inequality at almost all $t \in [0,T]$:*

$$\begin{aligned}\frac{1}{2}\|u\|^2(t) + \frac{1}{4\mu_2}\|\tau\|^2(t) + \int_0^t \frac{1}{2\lambda_1\mu_2}\|\tau\|^2\, ds + \int_0^t \mu_1\|u\|_Y^2\, ds\\ \le \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \int_0^t \langle f(s), u(s)\rangle\, ds.\end{aligned} \tag{6.6.2}$$

Proof. Consider the pairs (u_m, τ_m) which were used in the proof of Theorem 6.5.1 for the construction of the solution. Each of them satisfies inequality (6.6.1), and, therefore, (6.6.2). The rest of the proof (the passage to the limit in this inequality) is similar to the arguments in the proof of Lemma 6.6.1. □

6.6.2 Uniqueness of the weak solution

Theorem 6.6.1 (see [69]). *Assume that $n = 3$ and, under the conditions of Theorem 6.2.1, there is a weak solution (u_1, σ_1) to problem (6.2.1) – (6.2.5). If*

$$u_1 \in L_8(0,T;L_4(\Omega)^n), \quad \tau_1 = [\sigma_1 - 2\mu_1 \mathcal{E}(u_1)] \in L_4(0,T;\overset{\circ}{W}{}_4^1(\Omega,\mathbb{R}_S^{n\times n})), \tag{6.6.3}$$

then this solution is unique in the class of weak solutions (u,σ) (in the sense of Definition 6.2.1) which satisfy inequality (6.6.2) with $\tau = \sigma - 2\mu_1\mathcal{E}(u)$.

Proof. Let there be another weak solution (u_2,σ_2) to problem (6.2.1) – (6.2.5) satisfying inequality (6.6.2) with $\tau_2 = \sigma_2 - 2\mu_1\mathcal{E}(u_2)$. Let us show that it coincides with (u_1,σ_1), i.e. that (u_2,τ_2) coincides with (u_1,τ_1).

Obviously, both pairs (u_1,τ_1) and (u_2,τ_2) are solutions of problem (6.3.1), (6.3.2), (6.3.5).

As $u_1 \in L_8(0,T;L_4)$, one has $u_{1i}u_1 \in L_4(0,T;L_2)$ by inequality (2.2.1). Furthermore, since $u_1 \in L_\infty(0,T;L_2)$, $\tau_1 \in L_4(0,T;\overset{\circ}{W}{}_4^1(\Omega,\mathbb{R}_S^{n\times n}))$, $\overset{\circ}{W}{}_4^1(\Omega,\mathbb{R}_S^{n\times n}) \subset L_\infty(\Omega,\mathbb{R}_S^{n\times n})$, one has $u_{1i}\tau_1 \in L_4(0,T;L_2)$. Using Lemma 2.2.8 (cf. Remark 6.5.1) we conclude that (u_1,τ_1) satisfies (6.3.1), (6.3.2) at all $\varphi \in V$, $\Phi \in H_0^1$, and $\frac{du_1}{dt} \in L_2(0,T;V^*)$, $\frac{d\tau_1}{dt} \in L_2(0,T;H^{-1})$.

Put $\tau = \tau_1$, $\Phi = \frac{\tau_1(t)}{2\mu_2}$ in (6.3.1) and $u(t) = \varphi = u_1(t)$ in (6.3.2) at almost all $t \in [0,T]$, and add the results. Taking into account (6.1.18) – (6.1.20) we get:

$$\frac{1}{2}\frac{d}{dt}(u_1,u_1) + \frac{1}{4\mu_2}\frac{d}{dt}(\tau_1,\tau_1) + \mu_1(\nabla u_1,\nabla u_1) + \frac{1}{2\lambda_1\mu_2}(\tau_1,\tau_1) = \langle f,u_1\rangle. \tag{6.6.4}$$

Integrating this equality from 0 to arbitrary $t \in [0,T]$, we obtain the energy equality

$$\begin{aligned}\frac{1}{2}\|u_1\|^2(t) + \frac{1}{4\mu_2}\|\tau_1\|^2(t) + \int_0^t \frac{1}{2\lambda_1\mu_2}\|\tau_1\|^2\,ds + \int_0^t \mu_1\|u_1\|_Y^2\,ds \\ = \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \int_0^t \langle f(s),u_1(s)\rangle\,ds.\end{aligned} \tag{6.6.5}$$

Integration by parts in the third and fourth term of equality (6.3.1) with (u_1,τ_1) implies that (u_1,τ_1) satisfies the identity:

$$\frac{d}{dt}(\tau_1,\Phi) + \frac{1}{\lambda_1}(\tau_1,\Phi) + \sum_{i=1}^n \Big(u_{1i}\frac{\partial\tau_1}{\partial x_i},\Phi\Big) - 2\mu_2(\mathcal{E}(u_1),\Phi) = 0. \tag{6.6.6}$$

By (2.2.1), $u_{1i}\frac{\partial\tau_1}{\partial x_i} \in L_2(0,T;L_2)$. Then Lemma 2.2.8 yields that (6.6.6) is fulfilled for all $\Phi \in L_2$.

Take $\Phi = \dfrac{\tau_2(t)}{2\mu_2}$ in (6.6.6) and $u = u_1, \varphi = u_2(t)$ in (6.3.2) at almost all $t \in [0, T]$, and add the results (taking into account (6.3.7)):

$$\begin{aligned}
&\langle \frac{d}{dt} u_1, u_2 \rangle + \frac{1}{2\mu_2} \langle \frac{d}{dt} \tau_1, \tau_2 \rangle + \frac{1}{2\lambda_1 \mu_2} (\tau_1, \tau_2) + \frac{1}{2\mu_2} \sum_{i=1}^{n} \big(u_{1i} \frac{\partial \tau_1}{\partial x_i}, \tau_2 \big) \\
&\qquad - (\nabla u_1, \tau_2) - \sum_{i=1}^{n} \big(u_{1i} u_1, \frac{\partial u_2}{\partial x_i} \big) + \mu_1 (\nabla u_1, \nabla u_2) + (\tau_1, \nabla u_2) \\
&\quad = \langle f, u_2 \rangle.
\end{aligned} \tag{6.6.7}$$

Integrating by parts in the second term of equality (6.3.2) with (u_2, τ_2), we conclude that (u_2, τ_2) satisfies the identity:

$$\frac{d}{dt} (u_2, \varphi) + \sum_{i=1}^{n} \big(u_{2i} \frac{\partial u_2}{\partial x_i}, \varphi \big) + \mu_1 (\nabla u_2, \nabla \varphi) + (\tau_2, \nabla \varphi) = \langle f, \varphi \rangle. \tag{6.6.8}$$

Note that (6.6.8) holds true for all $\varphi \in V$ (see Remark 6.5.1).

Take $\tau = \tau_2, \Phi = \frac{\tau_1(t)}{2\mu_2}$ in (6.3.1) (we can do it due to Remark 6.5.1) and $\varphi = u_1(t)$ in (6.6.8) at almost all $t \in [0, T]$, and add the results, taking into account (6.3.7):

$$\begin{aligned}
&\langle \frac{d}{dt} u_2, u_1 \rangle + \frac{1}{2\mu_2} \langle \frac{d}{dt} \tau_2, \tau_1 \rangle + \frac{1}{2\lambda_1 \mu_2} (\tau_2, \tau_1) - \frac{1}{2\mu_2} \sum_{i=1}^{n} \big(u_{2i} \tau_2, \frac{\partial \tau_1}{\partial x_i} \big) \\
&\qquad - (\nabla u_2, \tau_1) + \mu_1 (\nabla u_2, \nabla u_1) + \sum_{i=1}^{n} \big(u_{2i} \frac{\partial u_2}{\partial x_i}, u_1 \big) + (\tau_2, \nabla u_1) \\
&\quad = \langle f, u_1 \rangle.
\end{aligned} \tag{6.6.9}$$

Inequality (2.1.25) implies

$$\|u_2\|_{L_{8/3}(0,T;L_4)} \le 2^{1/2} \|u_2\|^{\frac{1}{4}}_{L_\infty(0,T;L_2)} \|u_2\|^{\frac{3}{4}}_{L_2(0,T;V)} < +\infty.$$

But $\nabla u_2 \in L_2(0, T; L_2)$, so, by (2.2.1), $u_{2i} \frac{\partial u_2}{\partial x_i} \in L_{8/7}(0, T; L_{4/3})$. Then (6.6.8) yields $\frac{du_2}{dt} \in L_{8/7}(0, T; L_{4/3}) + L_2(0, T; V^*)$. Therefore

$$u_2 = u_{21} + u_{22}, \quad \frac{du_{21}}{dt} \in L_{8/7}(0, T; L_{4/3}), \quad \frac{du_{22}}{dt} \in L_2(0, T; V^*).$$

Similarly $u_{1i} \frac{\partial \tau_1}{\partial x_i} \in L_{8/3}(0, T; L_2)$ and $\frac{d\tau_1}{dt} \in L_2(0, T; L_2), u_{2i} \tau_2 \in L_{8/3}(0, T; L_{4/3})$ and $\frac{d\tau_2}{dt} \in L_2(0, T; W^{-1}_{4/3})$. Hence, all terms of equalities (6.6.7) and (6.6.9) are integrable on $(0, T)$.

Let us check the following formula:

$$\begin{aligned}\Big\langle\frac{d}{dt}u_2,u_1\Big\rangle+\frac{1}{2\mu_2}\Big\langle\frac{d}{dt}\tau_2,\tau_1\Big\rangle+\Big\langle\frac{d}{dt}u_1,u_2\Big\rangle+\frac{1}{2\mu_2}\Big\langle\frac{d}{dt}\tau_1,\tau_2\Big\rangle\\=\frac{d}{dt}\Big[(u_1,u_2)+\frac{1}{2\mu_2}(\tau_1,\tau_2)\Big].\end{aligned}\tag{6.6.10}$$

We are going to use the time averaging. Let

$$\psi_h(t)=\frac{1}{h}\int_{\frac{T-h}{T}t}^{\frac{T-h}{T}t+h}\psi(s)\,ds,$$

where ψ is a function of scalar argument with values in a Banach space X, and h is a small positive parameter (cf. (2.2.7)). It is easy to see that if $\psi\in L_p(0,T;X)$, then $\psi_h\in W_p^1(0,T;X)$, and $\psi_h\to\psi$ in $L_p(0,T;X)$ at $h\to 0$. Besides, this operation of averaging commutes with the derivation with respect to t. Therefore, if $\psi\in W_p^1(0,T;X)$, then $\psi_h\in W_p^2(0,T;X)$ and $\psi_h\to\psi$ in $W_p^1(0,T;X)$ at $h\to 0$.

Now, observe that

$$\begin{aligned}\Big\langle\frac{d}{dt}u_{2h},u_{1h}\Big\rangle+\frac{1}{2\mu_2}\Big\langle\frac{d}{dt}\tau_{2h},\tau_{1h}\Big\rangle+\Big\langle\frac{d}{dt}u_{1h},u_{2h}\Big\rangle+\frac{1}{2\mu_2}\Big\langle\frac{d}{dt}\tau_{1h},\tau_{2h}\Big\rangle\\=\frac{d}{dt}\Big[(u_{1h},u_{2h})+\frac{1}{2\mu_2}(\tau_{1h},\tau_{2h})\Big].\end{aligned}\tag{6.6.11}$$

We have that $u_{1h}\to u_1$ in $L_8(0,T;L_4)$ and in $L_2(0,T;V)$. Furthermore, $u_2=u_{21}+u_{22}$, $\frac{du_{21h}}{dt}\to\frac{du_{21}}{dt}$ in $L_{8/7}(0,T;L_{4/3})$, $\frac{du_{22h}}{dt}\to\frac{du_{22}}{dt}$ in $L_2(0,T;V^*)$. Therefore, almost all on $(0,T)$,

$$\begin{aligned}\Big\langle\frac{d}{dt}u_{2h},u_{1h}\Big\rangle=\Big\langle\frac{d}{dt}u_{21h},u_{1h}\Big\rangle+\Big\langle\frac{d}{dt}u_{22h},u_{1h}\Big\rangle\\\underset{h\to\infty}{\to}\Big\langle\frac{d}{dt}u_{21},u_1\Big\rangle+\Big\langle\frac{d}{dt}u_{22},u_1\Big\rangle=\Big\langle\frac{d}{dt}u_2,u_1\Big\rangle.\end{aligned}$$

Similarly, the remaining terms of the left-hand side of (6.6.11) converge to the corresponding terms of the left-hand side of (6.6.10). Besides,

$$(u_{1h},u_{2h})+\frac{1}{2\mu_2}(\tau_{1h},\tau_{2h})\to(u_1,u_2)+\frac{1}{2\mu_2}(\tau_1,\tau_2).$$

Passing to the limit as $h\to\infty$ in (6.6.11) in the sense distributions on $(0,T)$, we obtain (6.6.10).

Let $w=u_1-u_2,\sigma_*=\tau_1-\tau_2$. Observe that

$$\begin{aligned}-\sum_{i=1}^n\Big(u_{1i}u_1,\frac{\partial u_2}{\partial x_i}\Big)+\sum_{i=1}^n\Big(u_{2i}\frac{\partial u_2}{\partial x_i},u_1\Big)=-\sum_{i=1}^n\Big(w_i\frac{\partial u_2}{\partial x_i},u_1\Big)\\=\sum_{i=1}^n\Big(w_i\frac{\partial w}{\partial x_i},u_1\Big)-\sum_{i=1}^n\Big(w_i\frac{\partial u_1}{\partial x_i},u_1\Big).\end{aligned}$$

Due to (6.1.18) the last term vanishes. Thus,

$$-\sum_{i=1}^{n}\big(u_{1i}u_1, \frac{\partial u_2}{\partial x_i}\big) + \sum_{i=1}^{n}\big(u_{2i}\frac{\partial u_2}{\partial x_i}, u_1\big) = \sum_{i=1}^{n}\big(w_i\frac{\partial w}{\partial x_i}, u_1\big). \tag{6.6.12}$$

Besides,

$$\begin{aligned}\sum_{i=1}^{n}\big(u_{1i}\frac{\partial \tau_1}{\partial x_i}, \tau_2\big) - \sum_{i=1}^{n}\big(u_{2i}\tau_2, \frac{\partial \tau_1}{\partial x_i}\big) &= \sum_{i=1}^{n}\big(w_i\tau_2, \frac{\partial \tau_1}{\partial x_i}\big)\\ &= -\sum_{i=1}^{n}\big(w_i\sigma_*, \frac{\partial \tau_1}{\partial x_i}\big) + \sum_{i=1}^{n}\big(w_i\tau_1, \frac{\partial \tau_1}{\partial x_i}\big).\end{aligned}$$

By (6.1.19) the last term is zero. Thus,

$$\sum_{i=1}^{n}\big(u_{1i}\frac{\partial \tau_1}{\partial x_i}, \tau_2\big) - \sum_{i=1}^{n}\big(u_{2i}\tau_2, \frac{\partial \tau_1}{\partial x_i}\big) = -\sum_{i=1}^{n}\big(w_i\sigma_*, \frac{\partial \tau_1}{\partial x_i}\big). \tag{6.6.13}$$

Now, adding (6.6.7) and (6.6.9), integrating from 0 to any $t \in [0, T]$, and using (6.6.10), (6.6.12), and (6.6.13), we obtain the equality:

$$\begin{aligned}&(u_1, u_2)(t) + \frac{1}{2\mu_2}(\tau_1, \tau_2)(t) + \sum_{i=1}^{n}\int_0^t \big(w_i\frac{\partial w}{\partial x_i}, u_1\big)\, ds\\ &\qquad - \frac{1}{2\mu_2}\sum_{i=1}^{n}\int_0^t \big(w_i\sigma_*, \frac{\partial \tau_1}{\partial x_i}\big)\, ds + \int_0^t \frac{1}{\lambda_1\mu_2}(\tau_1, \tau_2)\, ds\\ &\qquad + 2\int_0^t \mu_1(u_1, u_2)_Y\, ds\\ &\quad = \|a\|^2 + \frac{1}{2\mu_2}\|\tau_0\|^2 + \int_0^t \langle f(s), u_1(s) + u_2(s)\rangle\, ds.\end{aligned} \tag{6.6.14}$$

But (u_2, τ_2) satisfies inequality (6.6.2):

$$\begin{aligned}&\frac{1}{2}\|u_2\|^2(t) + \frac{1}{4\mu_2}\|\tau_2\|^2(t) + \int_0^t \frac{1}{2\lambda_1\mu_2}\|\tau_2\|^2\, ds + \int_0^t \mu_1\|u_2\|_Y^2\, ds\\ &\quad \le \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \int_0^t \langle f(s), u_2(s)\rangle\, ds.\end{aligned} \tag{6.6.15}$$

Adding it with (6.6.4), and subtracting (6.6.14), we get:

$$\begin{aligned}&\frac{1}{2}\|w\|^2(t) + \frac{1}{4\mu_2}\|\sigma_*\|^2(t) + \int_0^t \frac{1}{2\lambda_1\mu_2}\|\sigma_*\|^2\, ds + \int_0^t \mu_1\|w\|_Y^2\, ds\\ &\quad \le \sum_{i=1}^{n}\int_0^t \big(w_i\frac{\partial w}{\partial x_i}, u_1\big)\, ds - \frac{1}{2\mu_2}\sum_{i=1}^{n}\int_0^t \big(w_i\sigma_*, \frac{\partial \tau_1}{\partial x_i}\big)\, ds.\end{aligned} \tag{6.6.16}$$

Applying Hölder's inequality, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\|w\|^2(t)+\frac{1}{4\mu_2}\|\sigma_*\|^2(t)+\int_0^t\frac{1}{2\lambda_1\mu_2}\|\sigma_*\|^2\,ds+\int_0^t\mu_1\|w\|_Y^2\,ds\\
&\qquad\le\int_0^t\big(\|w\|_{L_4}\|w\|_Y\|u_1\|_{L_4}+\frac{1}{2\mu_2}\|w\|_{L_4}\|\sigma_*\|\|\tau_1\|_{W_4^1}\big)\,ds.
\end{aligned}
\tag{6.6.17}
$$

We recall Young's inequality for scalars: for any $\varepsilon>0$ and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, there is a constant C such that for all $a,b\ge 0$ one has

$$ab\le\varepsilon a^p+Cb^q.$$

But, by (2.1.24),

$$\|w\|_{L_4}\|w\|_Y\|u_1\|_{L_4}\le 2^{1/2}\|w\|^{1/4}\|w\|_Y^{7/4}\|u_1\|_{L_4}.$$

Due to Young's inequality there is a constant K_1 such that the last expression does not exceed $\frac{1}{2}\mu_1\|w\|_Y^2+K_1\|w\|^2\|u_1\|_{L_4}^8$.

Furthermore, (2.1.24) and Young's inequality give

$$
\begin{aligned}
&\|w\|_{L_4}\|\sigma_*\|\|\tau_1\|_{W_4^1}\\
&\qquad\le\|w\|^{1/4}\|\sigma_*\|^{1/4}\|w\|_Y^{3/4}\|\sigma_*\|^{3/4}\|\tau_1\|_{W_4^1}\\
&\qquad\le\sqrt{\frac{\mu_1}{\lambda_1\mu_2}}\|w\|_Y\|\sigma_*\|+K_2\|w\|\|\sigma_*\|\|\tau_1\|_{W_4^1}^4\\
&\qquad\le\frac{1}{2}\mu_1\|w\|_Y^2+\frac{1}{2\lambda_1\mu_2}\|\sigma_*\|^2+\frac{1}{2}K_2(\|w\|^2+\|\sigma_*\|^2)\|\tau_1\|_{W_4^1}^4.
\end{aligned}
$$

Then (6.6.17) implies:

$$
\begin{aligned}
&\frac{1}{2}\|w\|^2(t)+\frac{1}{4\mu_2}\|\sigma_*\|^2(t)+\int_0^t\frac{1}{2\lambda_1\mu_2}\|\sigma_*\|^2\,ds+\int_0^t\mu_1\|w\|_Y^2\,ds\\
&\qquad\le\int_0^t(\mu_1\|w\|_Y^2+\frac{1}{2\lambda_1\mu_2}\|\sigma_*\|^2\\
&\qquad\qquad+\frac{1}{2}K_2(\|w\|^2+\|\sigma_*\|^2)\|\tau_1\|_{W_4^1}^4+K_1\|w\|^2\|u_1\|_{L_4}^8)\,ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{1}{2}\|w\|^2(t)+\frac{1}{4\mu_2}\|\sigma_*\|^2(t)\\
&\qquad\le\int_0^t\big(\frac{1}{2}K_2(\|w\|^2+\|\sigma_*\|^2)\|\tau_1\|_{W_4^1}^4+K_1\|w\|^2\|u_1\|_{L_4}^8\big)\,ds.
\end{aligned}
\tag{6.6.18}
$$

But the functions $\|u_1\|_{L_4}^8(t)$ and $\|\tau_1\|_{W_4^1}^4(t)$ are integrable on $(0,T)$. Then, by the Gronwall lemma, $w\equiv 0,\sigma_*\equiv 0$, and the theorem is proved. □

In a similar way one can prove another uniqueness result:

Theorem 6.6.2. *Assume that $n = 3$ and, under the conditions of Theorem 6.2.1, there are two weak solutions (u_1, σ_1), (u_2, σ_2) to problem (6.2.1) – (6.2.5). If*

$$\begin{gathered} u_j \in L_8(0, T; L_4(\Omega)^n), \\ \tau_j = [\sigma_j - 2\mu_1 \mathcal{E}(u_j)] \in L_4(0, T; W_4^1(\Omega, \mathbb{R}_S^{n\times n})), \quad j = 1, 2, \end{gathered} \tag{6.6.19}$$

then $u_1 = u_2, \sigma_1 = \sigma_2$.

Proof. Let $w = u_1 - u_2, \sigma_* = \tau_1 - \tau_2$. Note that (6.6.19) implies

$$\frac{d}{dt}(\tau_j, \Phi) + \frac{1}{\lambda_1}(\tau_j, \Phi) + \sum_{i=1}^{n} \Big(u_{ji} \frac{\partial \tau_j}{\partial x_i}, \Phi\Big) - 2\mu_2(\mathcal{E}(u_j), \Phi) = 0 \tag{6.6.20}$$

for all $\Phi \in L_2$ (cf. (6.6.6)). Furthermore, by Lemma 2.2.8, the pairs (u_j, τ_j) satisfy (6.3.2) for all $\varphi \in V$ almost all on $(0, T)$.

Put $\Phi = \dfrac{\tau_j(t)}{2\mu_2}$ in (6.6.20) and $u(t) = \varphi = u_j(t)$ in (6.3.2) at almost all $t \in [0, T]$, and add the results:

$$\frac{1}{2}\frac{d}{dt}(u_j, u_j) + \frac{1}{4\mu_2}\frac{d}{dt}(\tau_j, \tau_j) + \mu_1(\nabla u_j, \nabla u_j) + \frac{1}{2\lambda_1\mu_2}(\tau_j, \tau_j) = \langle f, u_j\rangle. \tag{6.6.21}$$

Integrating this equality from 0 to arbitrary $t \in [0, T]$, we obtain the energy equality (cf. (6.6.5)):

$$\begin{aligned} &\frac{1}{2}\|u_j\|^2(t) + \frac{1}{4\mu_2}\|\tau_j\|^2(t) + \int_0^t \frac{1}{2\lambda_1\mu_2}\|\tau_j\|^2\, ds + \int_0^t \mu_1 \|u_j\|_Y^2\, ds \\ &\qquad = \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \int_0^t \langle f(s), u_j(s)\rangle\, ds. \end{aligned} \tag{6.6.22}$$

Taking $j = 1, \Phi = \frac{\tau_2(t)}{2\mu_2}$ in (6.6.20) and $u = u_1, \varphi = u_2(t)$ in (6.3.2) at almost all $t \in [0, T]$, and adding the results, we get (6.6.7). Taking $j = 2, \Phi = \frac{\tau_1(t)}{2\mu_2}$ in (6.6.20) and $u = u_2, \varphi = u_1(t)$ in (6.3.2) at almost all $t \in [0, T]$, and adding the results, we get (6.6.9). From (6.6.7) and (6.6.9) we obtain (6.6.14).

Adding (6.6.22) for $j = 1, 2$, and subtracting (6.6.14), we get (6.6.16), and, hence, (6.6.18). $\square$

6.7 Minimal trajectory and global attractors for the Jeffreys model

6.7.1 Integral energy estimate: autonomous case

In Sections 6.7 and 6.8 we study the attractors for weak solutions of the Jeffreys model. In these two sections we assume that Ω is an arbitrary bounded domain in $\mathbb{R}^n$, $n = 2, 3$ (in this situation the spaces V and Y coincide up to equivalent norm). According to Remark 6.3.1, we investigate here problem (6.3.1), (6.3.2). First we have to derive an integral energy estimate for this problem.

Theorem 6.7.1 (see [76]). *Let $f \in V^*$. Given $a \in H$, $\tau_0 \in L_2(\Omega, \mathbb{R}_S^{n\times n})$, the solution (u, τ) to problem (6.3.1), (6.3.2), (6.3.5) constructed in Theorem 6.5.1 satisfies the energy inequality:*

$$\begin{aligned}\frac{1}{4}\|u\|^2_{L_\infty(t,t+1;H)} + \frac{1}{8\mu_2}\|\tau\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\|u\|^2_{L_2(t,t+1;Y)}\\ \le e^{-2\gamma t}\left(\|a\|^2 + \frac{1}{2\mu_2}\|\tau_0\|^2\right) + \frac{\gamma+1}{2\mu_1\gamma}\|f\|^2_{Y^*}\end{aligned} \tag{6.7.1}$$

for $t \in [0, T-1]$. Here $\gamma = \min(\frac{1}{\lambda_1}, \frac{\mu_1}{2K_0(\Omega)^2})$, where K_0 is the constant from Friedrichs' inequality (2.1.26).

Proof. Consider the pairs (u_k, τ_k), $k \in \mathbb{N}$ which were used in the proof of Theorem 6.5.1 for the construction of the solution. Each of them, in turn, was constructed in Theorem 6.4.1 using the sequences of solutions $(u_{m,k}, \tau_{m,k})$ to problem (6.3.3), (6.3.4) with $\delta = \frac{1}{m}$ and $\varepsilon = \frac{1}{k}$. It suffices to show that these solutions satisfy (6.7.1).

But these solutions $(u_{m,k}, \tau_{m,k})$ satisfy (6.3.8):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(u_{m,k}, u_{m,k}) + \frac{1}{4\mu_2}\frac{d}{dt}(\tau_{m,k}, \tau_{m,k}) + \mu_1(\nabla u_{m,k}, \nabla u_{m,k})\\ + \frac{1}{2\lambda_1\mu_2}(\tau_{m,k}, \tau_{m,k}) + \frac{1}{2k\lambda_1\mu_2}(\nabla\tau_{m,k}, \nabla\tau_{m,k}) = \langle f, u_{m,k}\rangle.\end{aligned} \tag{6.7.2}$$

Let $\overline{u}_{m,k}(t) = e^{\gamma t}u_{m,k}(t)$, $\overline{\tau}_{m,k}(t) = e^{\gamma t}\tau_{m,k}(t)$ where $t \in [0, T]$. Then we have:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}e^{-2\gamma t}\|\overline{u}_{m,k}\|^2 + \frac{1}{4\mu_2}\frac{d}{dt}e^{-2\gamma t}\|\overline{\tau}_{m,k}\|^2 + \mu_1 e^{-2\gamma t}\|\overline{u}_{m,k}\|^2_Y\\ + \frac{1}{2\lambda_1\mu_2}e^{-2\gamma t}\|\overline{\tau}_{m,k}\|^2 + \frac{1}{2k\lambda_1\mu_2}e^{-2\gamma t}\|\overline{\tau}_{m,k}\|^2_Y = \langle f, e^{-\gamma t}\overline{u}_{m,k}\rangle.\end{aligned}$$

Applying the formula of derivative of a product and multiplying by $e^{2\gamma t}$ we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\overline{u}_{m,k}\|^2-\gamma\|\overline{u}_{m,k}\|^2+\frac{1}{4\mu_2}\frac{d}{dt}\|\overline{\tau}_{m,k}\|^2-\frac{\gamma}{2\mu_2}\|\overline{\tau}_{m,k}\|^2\\&\quad+\mu_1\|\overline{u}_{m,k}\|_Y^2+\frac{1}{2\lambda_1\mu_2}\|\overline{\tau}_{m,k}\|^2+\frac{1}{2k\lambda_1\mu_2}\|\overline{\tau}_{m,k}\|_Y^2=\langle e^{\gamma t}f,\overline{u}_{m,k}\rangle.\end{aligned} \tag{6.7.3}$$

Since $\|\overline{u}_{m,k}\|\le K_0(\Omega)\|\overline{u}_{m,k}\|_Y$, (6.7.3) implies

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}_{m,k}\|^2+\frac{1}{4\mu_2}\frac{d}{dt}\|\overline{\tau}_{m,k}\|^2+\frac{\mu_1}{2}\|\overline{u}_{m,k}\|_Y^2\le e^{\gamma t}\|f\|_{Y^*}\|\overline{u}_{m,k}\|_Y.$$

Applying Cauchy's inequality we get

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}_{m,k}\|^2+\frac{1}{4\mu_2}\frac{d}{dt}\|\overline{\tau}_{m,k}\|^2+\frac{\mu_1}{2}\|\overline{u}_{m,k}\|_Y^2\le\frac{e^{2\gamma t}}{2\mu_1}\|f\|_{Y^*}^2+\frac{\mu_1}{2}\|\overline{u}_{m,k}\|_Y^2,$$

that is

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}_{m,k}\|^2+\frac{1}{4\mu_2}\frac{d}{dt}\|\overline{\tau}_{m,k}\|^2\le\frac{e^{2\gamma t}}{2\mu_1}\|f\|_{Y^*}^2.$$

Integrating from 0 to t we obtain

$$\frac{1}{2}\|\overline{u}_{m,k}\|^2+\frac{1}{4\mu_2}\|\overline{\tau}_{m,k}\|^2\le\frac{1}{2}\|a\|^2+\frac{1}{4\mu_2}\|\tau_0\|^2+\frac{1}{2\mu_1}\|f\|_{Y^*}^2\int_0^t e^{2\gamma s}\,ds.$$

This implies

$$\frac{1}{2}\|e^{\gamma t}u_{m,k}\|^2+\frac{1}{4\mu_2}\|e^{\gamma t}\tau_{m,k}\|^2\le\frac{1}{2}\|a\|^2+\frac{1}{4\mu_2}\|\tau_0\|^2+\frac{1}{4\gamma\mu_1}\|f\|_{Y^*}^2(e^{2\gamma t}-1).$$

Multiplying by $2e^{-2\gamma t}$ and taking the maximum along the interval $(t,t+1)$ we conclude

$$\begin{aligned}&\max_{s\in(t,t+1)}\Big(\|u_{m,k}(s)\|^2+\frac{1}{2\mu_2}\|\tau_{m,k}(s)\|^2\Big)\\&\qquad\le e^{-2\gamma t}\big(\|a\|^2+\frac{1}{2\mu_2}\|\tau_0\|^2\big)+\frac{1}{2\gamma\mu_1}\|f\|_{Y^*}^2.\end{aligned} \tag{6.7.4}$$

Applying to the right-hand side of (6.7.2) Cauchy's inequality we get:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_{m,k}\|^2+\frac{1}{4\mu_2}\frac{d}{dt}\|\tau_{m,k}\|^2+\mu_1\|u_{m,k}\|_Y^2+\frac{1}{2\lambda_1\mu_2}\|\tau_{m,k}\|^2\\&\quad+\frac{1}{2k\lambda_1\mu_2}\|\tau_{m,k}\|_Y^2\le\frac{1}{2\mu_1}\|f\|_{Y^*}^2+\frac{\mu_1}{2}\|u_{m,k}\|_Y^2.\end{aligned}$$

Hence,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_{m,k}\|^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\tau_{m,k}\|^2 + \frac{\mu_1}{2}\|u_{m,k}\|_Y^2 + \frac{1}{2k\lambda_1\mu_2}\|\tau_{m,k}\|_Y^2 \\
&\quad \le \frac{1}{2\mu_1}\|f\|_{Y^*}^2.
\end{aligned}
$$

Integrating from t to $t+1$ we obtain

$$
\begin{aligned}
&\frac{1}{2}\|u_{m,k}(t+1)\|^2 - \frac{1}{2}\|u_{m,k}(t)\|^2 + \frac{1}{4\mu_2}\|\tau_{m,k}(t+1)\|^2 - \frac{1}{4\mu_2}\|\tau_{m,k}(t)\|^2 \\
&\quad + \frac{\mu_1}{2}\int_t^{t+1}\|u_{m,k}(s)\|_Y^2\,ds + \frac{1}{2k\lambda_1\mu_2}\int_t^{t+1}\|\tau_{m,k}(s)\|_Y^2\,ds \le \frac{1}{2\mu_1}\|f\|_{Y^*}^2.
\end{aligned} \tag{6.7.5}
$$

Adding (6.7.4) and (6.7.5) we get:

$$
\begin{aligned}
&\frac{1}{2}\|u_{m,k}(t+1)\|^2 + \frac{1}{4\mu_2}\|\tau_{m,k}(t+1)\|^2 \\
&\quad + \max_{s\in(t,t+1)}\Big(\frac{1}{2}\|u_{m,k}(s)\|^2 + \frac{1}{4\mu_2}\|\tau_{m,k}(s)\|^2\Big) \\
&\quad + \frac{\mu_1}{2}\int_t^{t+1}\|u_{m,k}(s)\|_Y^2\,ds + \frac{1}{2k\lambda_1\mu_2}\int_t^{t+1}\|\tau_{m,k}(s)\|_Y^2\,ds \\
&\le e^{-2\gamma t}\big(\|a\|^2 + \frac{1}{2\mu_2}\|\tau_0\|^2\big) + \frac{1+\gamma}{2\gamma\mu_1}\|f\|_{Y^*}^2, \quad 0 \le t \le T-1.
\end{aligned} \tag{6.7.6}
$$

Taking into account inequality (6.3.9) and ignoring the first two positive terms in the left-hand side we obtain

$$
\begin{aligned}
&\frac{1}{4}\max_{s\in(t,t+1)}\|u_{m,k}(s)\|^2 + \frac{1}{8\mu_2}\max_{s\in(t,t+1)}\|\tau_{m,k}(s)\|^2 \\
&\quad + \frac{\mu_1}{2}\int_t^{t+1}\|u_{m,k}(s)\|_Y^2\,ds + \frac{1}{2k\lambda_1\mu_2}\int_t^{t+1}\|\tau_{m,k}(s)\|_Y^2\,ds \\
&\le e^{-2\gamma t}\big(\|a\|^2 + \frac{1}{2\mu_2}\|\tau_0\|^2\big) + \frac{1+\gamma}{2\gamma\mu_1}\|f\|_{Y^*}^2,\ 0 \le t \le T-1,
\end{aligned} \tag{6.7.7}
$$

which is even stronger than (6.7.1). □

Let us also state the following

Lemma 6.7.1. *Let $f \in V^*$. Assume that a pair (u,τ), where*

$$u \in L_2(0,T;V)\bigcap L_\infty(0,T;H),\ \ \tau \in L_\infty(0,T;L_2), \tag{6.7.8}$$

satisfies either identities (6.3.1), (6.3.2) or identities (6.3.3), (6.3.4) almost everywhere on $(0,T)$ *for all* $\varphi \in \mathcal{V}$ *and* $\Phi \in C_0^\infty$*. Then for all* $t \in [0,T]$ *the following estimate is valid:*

$$\begin{aligned}&\|u'\|_{L_{4/3}(t,T;Y^*)} + \|\tau'\|_{L_2(t,T;H^{-2})}\\ &\quad \le K_1(\|u\|_{L_\infty(t,T;H)}, \|u\|_{L_2(t,T;Y)}, \|\tau\|_{L_\infty(t,T;L_2)}, \|f\|_{Y^*}, T-t, \Omega).\end{aligned}$$

A more general variant of this lemma will be proved in Section 6.8.

6.7.2 Existence and structure of attractors

In this subsection we construct the minimal trajectory attractor and the global attractor for problem (6.3.1), (6.3.2) in the autonomous case ($f \in V^*$). We are going to apply the abstract results of Section 4.2. Let us choose $H \times L_2(\Omega, \mathbb{R}_S^{n\times n})$ as the space E and the space $V_\delta^* \times H^{-\delta}(\Omega, \mathbb{R}_S^{n\times n})$ as the space E_0, where $\delta \in (0,1]$ is a fixed number. We have also to define the trajectory space $\mathcal{H}^+$ for the Jeffreys model. This will be the set of pairs of functions (u,τ) which

i) belong to the class

$$u \in L_{2,\mathrm{loc}}(0,+\infty;V) \bigcap L_\infty(0,+\infty;H), \;\; \tau \in L_\infty(0,+\infty;L_2); \tag{6.7.9}$$

ii) satisfy identities (6.3.1), (6.3.2) for almost everywhere $t \in (0,+\infty)$ and for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$;

iii) satisfy the energy inequality:

$$\begin{aligned}&\frac{1}{4}\|u\|^2_{L_\infty(t,t+1;H)} + \frac{1}{8\mu_2}\|\tau\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\|u\|^2_{L_2(t,t+1;Y)}\\ &\quad \le e^{-2\gamma t}\big(\|u\|^2_{L_\infty(0,+\infty;H)} + \frac{1}{2\mu_2}\|\tau\|^2_{L_\infty(0,+\infty;L_2)}\big) + \frac{\gamma+1}{2\mu_1\gamma}\|f\|^2_{Y^*}\end{aligned} \tag{6.7.10}$$

for all $t \ge 0$ where γ is as in Theorem 6.7.1.

Remark 6.7.1. In Section 4.2 it was supposed that $\mathcal{H}^+ \subset C([0,+\infty);E_0) \cap L_\infty(0,+\infty;E)$. Let us show that this condition holds for the Jeffreys model. In fact, every pair (u,τ) from $\mathcal{H}^+$ belongs to $L_\infty(0,+\infty;H\times L_2) = L_\infty(0,+\infty;E)$. For any $T \ge 0$ the function $u \in L_2(0,T;V)$. By Lemma 6.7.1 $u' \in L_{4/3}(0,T;V^*)$, $\tau' \in L_2(0,T;H^{-2})$. Since Ω is bounded, $H_0^\delta \subset L_2$ compactly. Therefore $V_\delta \subset H$ compactly, $L_2 \subset H^{-\delta}$ compactly, $H \subset V_\delta^*$ compactly. By Theorem 2.2.6 one has: $u \in C([0,T];V_\delta^*)$, $\tau \in C([0,T];H^{-\delta})$, that is $(u,\tau) \in C([0,T];E_0)$ for any $T \ge 0$.

Remark 6.7.2. On account of inequality (6.7.10) the trajectory space $\mathcal{H}^+$ for the Jeffreys model is not invariant with respect to the translation operator $T(h)$.

Theorem 6.7.2 (see [75, 76]). *Let $f \in V^*$. Given $a \in H$, $\tau_0 \in L_2(\Omega, \mathbb{R}_S^{n\times n})$, there is a pair of functions (a trajectory) $(u, \tau) \in \mathcal{H}^+$ satisfying initial condition (6.3.5).*

Proof. Take an increasing sequence of positive numbers $T_m \to \infty$. By Theorem 6.5.1, for every natural m there is a pair (u_m, τ_m) in class (6.5.1) which satisfies (6.3.1), (6.3.2) almost everywhere in $(0, T_m)$ as well as the initial condition (6.3.5). Denote by $\widetilde{u}_m$ and $\widetilde{\tau}_m$ the functions which are equal to u_m and τ_m in $[0, T_m]$ and are equal to zero on $(T_m, +\infty)$. Since, by Theorem 6.7.1, all pairs (u_m, τ_m) satisfy inequality (6.7.1), the sequences $\widetilde{u}_m$ and $\widetilde{\tau}_m$ are bounded in $L_\infty(0, +\infty; H)$ and $L_\infty(0, +\infty; L_2)$, respectively. Thus, without loss of generality one may assume that there exist limits

$$u = \lim_{m\to\infty} \widetilde{u}_m, \quad \text{which is } *\text{-weak in} L_\infty(0, +\infty; H);$$

$$\tau = \lim_{m\to\infty} \widetilde{\tau}_m, \quad \text{which is } *\text{-weak in} L_\infty(0, +\infty; L_2).$$

Fix an arbitrary interval $[0, T]$. Then estimate (6.7.1) for all pairs (u_m, τ_m) implies the uniform boundedness of the sequence $\{\widetilde{u}_m\}$ in $L_2(0, T; V)$. Therefore without loss of generality we may consider that $\widetilde{u}_m \rightharpoonup u$ weakly in $L_2(0, T; V)$. It is easy to see that the pair (u, τ) satisfies inequality (6.7.1) (and, hence, (6.7.10)) and for $T_m \geq T$ the functions $\widetilde{u}_m$ and $\widetilde{\tau}_m$ coincide with u_m and τ_m on the segment $[0, T]$.

Just as in the proof of Theorems 6.4.1 and 6.5.1, estimate (6.7.1) for all pairs (u_m, τ_m) together with Lemma 6.7.1 ensure the convergence of all terms in identities (6.3.1), (6.3.2) with (u_m, τ_m) substituted there (m should be sufficiently large: so that $T_m \geq T$) to the corresponding terms in (6.3.1), (6.3.2) with (u, τ) in the sense of scalar distributions on $(0, T)$ as $m \to \infty$. Therefore the pair (u, τ) satisfies identities (6.3.1), (6.3.2) almost everywhere on $(0, T)$ for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$, belongs to class (6.7.8) and satisfies condition (6.3.5). Since T was arbitrary, the pair (u, τ) satisfies conditions i) – iii) of the definition of $\mathcal{H}^+$. □

The main result of this section is

Theorem 6.7.3 (see [75, 76]). *Let $f \in V^*$. There exists a minimal trajectory attractor $\mathfrak{U}_J$ for the trajectory space $\mathcal{H}^+$ and (4.2.6) is fulfilled.*

Proof. It suffices to show existence of a trajectory semiattractor and to apply Theorems 4.2.1 and 4.2.5.

Consider the set P which consists of pairs

$$(u, \tau) \in C([0, +\infty); E_0) \cap L_\infty(0, +\infty; E)$$

satisfying inequalities

$$\frac{1}{4}\|u\|^2_{L_\infty(t,t+1;H)} + \frac{1}{8\mu_2}\|\tau\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\|u\|^2_{L_2(t,t+1;Y)} \le \frac{\gamma+1}{\mu_1\gamma}\|f\|^2_{Y^*},$$

$$\|u'\|_{L_{4/3}(t,t+1;Y^*)} + \|\tau'\|_{L_2(t,t+1;H^{-2})}$$
$$\le K_1(\|u\|_{L_\infty(t,t+1;H)}, \|u\|_{L_2(t,t+1;Y)}, \|\tau\|_{L_\infty(t,t+1;L_2)}, \|f\|_{Y^*}, 1, \Omega)$$

for all $t \ge 0$, where K_1 is the constant from Lemma 6.7.1.

The set P is bounded in $L_\infty(0,+\infty;E)$. Let us show that P is relatively compact in $C([0,+\infty);E_0)$.

Fix $\varepsilon > 0$. Let a natural number M be such that $\frac{1}{2^{M-1}} < \varepsilon$. By Theorem 2.2.6 the set $P_M = \{v = u|_{[0,M]} : u \in P\}$ is relatively compact in $C([0,M];E_0)$. Then there is an $\frac{\varepsilon}{4}$-net $\{u_1,\dots,u_k\} \subset P_M$ in $C([0,M];E_0)$ for the set P_M. Let $\widetilde{u}_j(t) = u_j(t), 0 \le t \le M, \widetilde{u}_j(t) = u_j(M), t > M$, $j = 1,\dots,k$. Then for any $u \in P$:

$$\min_{j=1,\dots,k}\|u-\widetilde{u}_j\|_{C([0,+\infty);E_0)} = \min_{j=1,\dots,k}\sum_{i=0}^{+\infty}2^{-i}\frac{\|u-\widetilde{u}_j\|_{C([0,i];E_0)}}{1+\|u-\widetilde{u}_j\|_{C([0,i];E_0)}}$$
$$\le \min_{j=1,\dots,k}\sum_{i=0}^{M}2^{-i}\|u-\widetilde{u}_j\|_{C([0,i];E_0)} + \sum_{i=M+1}^{+\infty}2^{-i} \le 2\cdot\frac{\varepsilon}{4}+\frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\{\widetilde{u}_1,\dots,\widetilde{u}_k\}$ is ε-net for P. Hence, P is relatively compact in $C([0,+\infty);E_0)$.

By (6.7.10) and Lemma 6.7.1 the set P is absorbing for the trajectory space $\mathcal{H}^+$. Furthermore, it is clear that $T(h)P \subset P$ for all $h \ge 0$. By Lemma 4.2.9, $\overline{P}$ (its closure in $C([0,+\infty);E_0)$) is a semiattractor for the trajectory space $\mathcal{H}^+$. □

Remark 6.7.3. Since the set P (and, hence, $\overline{P}$) is absorbing, we could apply Corollary 4.2.1 instead of Theorem 4.2.1.

Theorems 4.2.2 and 6.7.3 imply

Theorem 6.7.4. *Let $f \in V^*$. In the space $H \times L_2$ there is a global attractor $\mathcal{A}_J$ for problem (6.3.1) – (6.3.2), i.e. a minimal compact in $V_\delta^* \times H^{-\delta}$ and bounded in $H \times L_2$ set, which attracts all trajectories from $\mathcal{H}^+$ (see Definition 4.2.6). For all $t \ge 0$ we have*

$$\mathcal{K}(\mathcal{H}^+)(t) \subset \mathcal{A}_J = \mathcal{U}_J(t) = \mathcal{K}(\mathcal{U}_J)(t).$$

Remark 6.7.4. We have established existence of minimal trajectory and global attractors for the space $\mathcal{H}^+$ of solutions for problem (6.3.1) – (6.3.2) on the positive axis, which satisfy the integral energy inequality. At the same time it is not known whether there exist weak solutions of this problem which do not satisfy the energy inequality. Such a problem is open even for the Navier–Stokes system.

6.8 Uniform attractors for the Jeffreys model

6.8.1 Integral energy estimate: non-autonomous case

In this section we are going to investigate attractors of problem (6.3.1), (6.3.2) in the case when the body force depends on time. First let us introduce the space $\mathfrak{X}$ which consists of all elements from $L_{2,\text{loc}}(0,+\infty;V^*)$ for which the norm

$$\|\phi\|_{\mathfrak{X}} = \sup_{t\geq 0}\|\phi\|_{L_2(t,t+1;Y^*)}$$

is finite.

We have to derive an integral energy estimate in the non-autonomous case.

Theorem 6.8.1 (see [77]). *Let $f\in\mathfrak{X}$. Given $a\in H$, $\tau_0\in L_2(\Omega,\mathbb{R}_S^{n\times n})$, the solution (u,τ) to problem (6.3.1), (6.3.2), (6.3.5) constructed in Theorem 6.5.1 satisfies the energy inequality:*

$$\begin{aligned}\frac{1}{4}\|u\|^2_{L_\infty(t,t+1;H)} + \frac{1}{8\mu_2}\|\tau\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\|u\|^2_{L_2(t,t+1;Y)}&\\ \leq e^{-2\gamma t}\big(\|a\|^2 + \frac{1}{2\mu_2}\|\tau_0\|^2\big) + \frac{2e^{4\gamma}+e^{2\gamma}-1}{2\mu_1(e^{2\gamma}-1)}\|f\|^2_{\mathfrak{X}}&\end{aligned}\tag{6.8.1}$$

for $t\in[0,T-1]$ where γ is as in Theorem 6.7.1.

Proof. Consider the pairs (u_k,τ_k), $k\in\mathbb{N}$ which were used in the proof of Theorem 6.5.1 for the construction of the solution. Each of them, in turn, was constructed in Theorem 6.4.1 using the sequences of solutions $(u_{m,k},\tau_{m,k})$ to problem (6.3.3), (6.3.4) with $\delta=\frac{1}{m}$ and $\varepsilon=\frac{1}{k}$. It suffices to show that these solutions satisfy (6.8.1).

As in the proof of Theorem 6.7.1, these solutions $(u_{m,k},\tau_{m,k})$ satisfy (6.7.2). Let $\overline{u}_{m,k}(t)=e^{\gamma t}u_{m,k}(t)$, $\overline{\tau}_{m,k}(t)=e^{\gamma t}\tau_{m,k}(t)$ where $t\in[0,T]$. Then, as in Theorem 6.7.1, we have:

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}_{m,k}\|^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\overline{\tau}_{m,k}\|^2 \leq \frac{e^{2\gamma t}}{2\mu_1}\|f(t)\|^2_{Y^*}.$$

Integrating from 0 to t we obtain

$$\frac{1}{2}\|\overline{u}_{m,k}\|^2 + \frac{1}{4\mu_2}\|\overline{\tau}_{m,k}\|^2 \leq \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \frac{1}{2\mu_1}\int_0^t e^{2\gamma s}\|f(s)\|^2_{Y^*}\,ds.\tag{6.8.2}$$

Let us use the following simple inequality (see its proof after the proof of this theorem)

$$\int_0^t a^s\phi(s)\,ds \leq \frac{a^{t+2}}{a-1}\sup_{s\in[0,t-1]}\int_s^{s+1}|\phi(\zeta)|\,d\zeta,\tag{6.8.3}$$

where $a \geq 1$ and $\phi(s)$ is a scalar function. Then from (6.8.2) we get:

$$\begin{aligned}&\frac{1}{2}\|e^{\gamma t}u_{m,k}\|^2 + \frac{1}{4\mu_2}\|e^{\gamma t}\tau_{m,k}\|^2\\&\qquad\leq \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \frac{e^{2\gamma(t+2)}}{2\mu_1(e^{2\gamma}-1)}\sup_{s\in[0,t-1]}\int_s^{s+1}\|f(\zeta)\|^2_{Y^*}\,d\zeta.\end{aligned}$$

Multiplying by $2e^{-2\gamma t}$ and taking the maximum along the interval $(t, t+1)$ we conclude

$$\begin{aligned}&\max_{s\in(t,t+1)}\left(\|u_{m,k}(s)\|^2 + \frac{1}{2\mu_2}\|\tau_{m,k}(s)\|^2\right)\\&\qquad\leq e^{-2\gamma t}\left(\|a\|^2 + \frac{1}{2\mu_2}\|\tau_0\|^2\right) + \frac{e^{4\gamma}}{\mu_1(e^{2\gamma}-1)}\|f\|^2_{\mathfrak{X}}.\end{aligned}\tag{6.8.4}$$

As in the proof of Theorem 6.7.1, applying to the right-hand side of (6.7.2) Cauchy's inequality we get:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_{m,k}\|^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\tau_{m,k}\|^2 + \frac{\mu_1}{2}\|u_{m,k}\|^2_Y + \frac{1}{2k\lambda_1\mu_2}\|\tau_{m,k}\|^2_Y\\&\qquad\leq \frac{1}{2\mu_1}\|f(t)\|^2_{Y^*}.\end{aligned}$$

Integrating from t to $t+1$ we obtain

$$\begin{aligned}&\frac{1}{2}\|u_{m,k}(t+1)\|^2 - \frac{1}{2}\|u_{m,k}(t)\|^2 + \frac{1}{4\mu_2}\|\tau_{m,k}(t+1)\|^2 - \frac{1}{4\mu_2}\|\tau_{m,k}(t)\|^2\\&\quad+\frac{\mu_1}{2}\int_t^{t+1}\|u_{m,k}(s)\|^2_Y\,ds + \frac{1}{2k\lambda_1\mu_2}\int_t^{t+1}\|\tau_{m,k}(s)\|^2_Y\,ds \leq \frac{1}{2\mu_1}\|f\|^2_{\mathfrak{X}}.\end{aligned}\tag{6.8.5}$$

Adding (6.8.4) and (6.8.5) we get:

$$\begin{aligned}&\frac{1}{2}\|u_{m,k}(t+1)\|^2 + \frac{1}{4\mu_2}\|\tau_{m,k}(t+1)\|^2\\&\qquad+\max_{s\in(t,t+1)}\left(\frac{1}{2}\|u_{m,k}(s)\|^2 + \frac{1}{4\mu_2}\|\tau_{m,k}(s)\|^2\right)\\&\qquad+\frac{\mu_1}{2}\int_t^{t+1}\|u_{m,k}(s)\|^2_Y\,ds + \frac{1}{2k\lambda_1\mu_2}\int_t^{t+1}\|\tau_{m,k}(s)\|^2_Y\,ds\\&\quad\leq e^{-2\gamma t}\left(\|a\|^2 + \frac{1}{2\mu_2}\|\tau_0\|^2\right) + \frac{2e^{4\gamma}+e^{2\gamma}-1}{2\mu_1(e^{2\gamma}-1)}\|f\|^2_{\mathfrak{X}},\end{aligned}\tag{6.8.6}$$

$0 \leq t \leq T-1.$

Taking into account inequality (6.3.9) and ignoring the first two positive terms in the left-hand side we obtain

$$
\begin{aligned}
&\frac{1}{4}\max_{s\in(t,t+1)}\|u_{m,k}(s)\|^2+\frac{1}{8\mu_2}\max_{s\in(t,t+1)}\|\tau_{m,k}(s)\|^2\\
&\qquad+\frac{\mu_1}{2}\int_t^{t+1}\|u_{m,k}(s)\|_Y^2\,ds+\frac{1}{2k\lambda_1\mu_2}\int_t^{t+1}\|\tau_{m,k}(s)\|_Y^2\,ds\\
&\quad\le e^{-2\gamma t}\big(\|a\|^2+\frac{1}{2\mu_2}\|\tau_0\|^2\big)+\frac{2e^{4\gamma}+e^{2\gamma}-1}{2\mu_1(e^{2\gamma}-1)}\|f\|_{\mathfrak{X}}^2,\\
&0\le t\le T-1,
\end{aligned}
\tag{6.8.7}
$$

which yields (6.8.1). □

Proof of inequality (6.8.3). Without loss of generality we may assume that $\phi(s)=0$ for $s>t$. Denote by $[t]$ the integer part of t. Then we have:

$$
\begin{aligned}
\int_0^t a^s\phi(s)\,ds &\le \sum_{i=1}^{[t]}\int_i^{i+1}a^s|\phi(s)|\,ds\le\sum_{i=0}^{[t]}a^{i+1}\int_i^{i+1}|\phi(s)|\,ds\\
&\le a(1+a+a^2+\cdots+a^{[t]})\sup_{s\in[0,t]}\int_s^{s+1}|\phi(\zeta)|\,d\zeta\\
&=\frac{a(a^{[t]+1}-1)}{a-1}\sup_{s\in[0,t-1]}\int_s^{s+1}|\phi(\zeta)|\,d\zeta\\
&\le\frac{a^{t+2}}{a-1}\sup_{s\in[0,t-1]}\int_s^{s+1}|\phi(\zeta)|\,d\zeta.
\end{aligned}
$$

□

Now we turn to the generalization of Lemma 6.7.1.

Lemma 6.8.1. *Let $f\in\mathfrak{X}$. Assume that a pair (u,τ), where*

$$
u\in L_2(0,T;V)\bigcap L_\infty(0,T;H),\ \tau\in L_\infty(0,T;L_2),
\tag{6.8.8}
$$

satisfies either identities (6.3.1), (6.3.2) or identities (6.3.3), (6.3.4) almost everywhere on $(0,T)$ for all $\varphi\in\mathcal{V}$ and $\Phi\in C_0^\infty$. Then for all $t\in[0,T]$ the following estimate is valid:

$$
\begin{aligned}
&\|u'\|_{L_{4/3}(t,T;Y^*)}+\|\tau'\|_{L_2(t,T;H^{-2})}\\
&\quad\le K_1(\|u\|_{L_\infty(t,T;H)},\|u\|_{L_2(t,T;Y)},\|\tau\|_{L_\infty(t,T;L_2)},\|f\|_{\mathfrak{X}},T-t,\Omega).
\end{aligned}
\tag{6.8.9}
$$

Proof. Let a pair (u, τ) from class (6.8.8) satisfy identities (6.3.3), (6.3.4) almost everywhere on $(0, T)$ for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$. Inequalities (2.1.23), (2.1.24) and (2.1.26) imply that there is a constant $K_2 = K_2(\Omega)$ such that

$$\|u(t)\|_{L_4} \leq K_2(\Omega)\|u(t)\|^{\frac{1}{4}}\|u(t)\|_Y^{\frac{3}{4}}, \quad n = 2, 3. \tag{6.8.10}$$

Thus,

$$\|u\|_{L_{8/3}(t,T;L_4)} = \|\|u\|_{L_4(\Omega)}\|_{L_{8/3}(t,T)} \leq K_2(\Omega)\big\|\|u(t)\|^{\frac{1}{4}}\|u(t)\|_Y^{\frac{3}{4}}\big\|_{L_{8/3}(t,T)}.$$

By Hölder's inequality (2.1.1) with $\Omega = (0, T)$, $\psi_1 = \|u(t)\|^{\frac{1}{4}}$, $\psi_2 = \|u(t)\|_Y^{\frac{3}{4}}$, $p = p_2 = 8/3$, $p_1 = \infty$ the last expression does not exceed

$$K_2(\Omega)\|\|u(t)\|^{\frac{1}{4}}\|_{L_\infty(t,T)}\|\|u(t)\|_Y^{\frac{3}{4}}\|_{L_{8/3}(t,T)} = K_2(\Omega)\|u\|^{\frac{1}{4}}_{L_\infty(t,T;H)}\|u\|^{\frac{3}{4}}_{L_2(t,T;Y)}.$$

We have from (6.3.4):

$$\begin{aligned}
\|\langle u', \varphi\rangle\|_{L_{4/3}(t,T)} &= \Big\|\frac{d}{dt}(u, \varphi)\Big\|_{L_{4/3}(t,T)} \\
&\leq \sum_{i=1}^{n}\Big\|\Big(\frac{u_i u}{1 + \frac{1}{m}\Big(\frac{|\tau|^2}{2\mu_2} + |u|^2\Big)}, \frac{\partial\varphi}{\partial x_i}\Big)\Big\|_{L_{4/3}(t,T)} \\
&\quad + \|\mu_1(\nabla u, \nabla\varphi)\|_{L_{4/3}(t,T)} + \|(\tau, \nabla\varphi)\|_{L_{4/3}(t,T)} \\
&\quad + \|\langle f, \varphi\rangle\|_{L_{4/3}(t,T)}.
\end{aligned}$$

Applying Hölder's inequality (2.1.1), Lemma 2.2.1 a), and the inequality

$$\frac{1}{1 + \frac{1}{m}(\frac{|\tau|^2}{2\mu_2} + |u|^2)} \leq 1$$

we get that the right-hand side does not exceed

$$\begin{aligned}
&\|\varphi\|_Y\big(\|u\|^2_{L_{8/3}(t,T;L_4)} + \mu_1\|u\|_{L_{4/3}(t,T;Y)} + \|\tau\|_{L_{4/3}(t,T;L_2)} + \|f\|_{L_{4/3}(t,T;Y^*)}\big) \\
&\quad \leq \|\varphi\|_Y(K_2^2(\Omega)\|u\|^{1/2}_{L_\infty(t,T;H)}\|u\|^{\frac{3}{2}}_{L_2(t,T;Y)} + \mu_1\|u\|_{L_{4/3}(t,T;Y)} \\
&\qquad + \|\tau\|_{L_{4/3}(t,T;L_2)} + \|f\|_{L_{4/3}(t,T;Y^*)}) \\
&\quad \leq \|\varphi\|_Y K_3\big(\|u\|_{L_\infty(t,T;L_2)}, \|u\|_{L_2(t,T;Y)}, \|\tau\|_{L_\infty(t,T;L_2)}, \|f\|_{\mathfrak{X}}, T - t, \Omega\big),
\end{aligned}$$

and this gives the estimate of the first term in the left-hand side of (6.8.9).

Taking into account the embedding $H_0^2 \subset W_4^1$ for $n = 2, 3$, from (6.3.3) we obtain the estimate of the second term in the left-hand side of (6.8.9):

$$\begin{aligned}
\|\langle \tau', \Phi\rangle\|_{L_2(t,T)} &= \Big\| \frac{d}{dt}(\tau, \Phi)\Big\|_{L_2(t,T)} \\
&\le \Big\| \frac{1}{\lambda_1}(\tau, \Phi)\Big\|_{L_2(t,T)} + \frac{1}{k\lambda_1}\|(\tau, \Delta\Phi)\|_{L_2(t,T)} \\
&\quad + \Big\| \sum_{i=1}^{n} \Big(\frac{u_i \tau}{1 + \frac{1}{m}\Big(\frac{|\tau|^2}{2\mu_2} + |u|^2\Big)}, \frac{\partial \Phi}{\partial x_i}\Big)\Big\|_{L_2(t,T)} + \|2\mu_2(u, \operatorname{Div}\Phi)\|_{L_2(t,T)} \\
&\le \frac{1}{\lambda_1}\|\tau\|_{L_2(t,T;L_2)}\|\Phi\|_{L_2} + \frac{1}{k\lambda_1}\|\Phi\|_{H_0^2}\|\tau\|_{L_2(t,T;L_2)} \\
&\quad + \|u\|_{L_2(t,T;L_4)}\|\tau\|_{L_\infty(t,T;L_2)}\Big\|\frac{\partial \Phi}{\partial x_i}\Big\|_{L_4} + 2\mu_2\|u\|_{L_2(t,T;L_2)}\|\operatorname{Div}\Phi\|_{L_2} \\
&\le \|\Phi\|_{H_0^2} K_4\big(\|u\|_{L_\infty(t,T;L_2)}, \|u\|_{L_2(t,T;Y)}, \|\tau\|_{L_\infty(t,T;L_2)}, T - t, \Omega\big).
\end{aligned}$$

If (u, τ) satisfies identities (6.3.1), (6.3.2) instead of (6.3.3), (6.3.4), then the proof of (6.8.9) is carried out in the same way. □

Remark 6.8.1. For arbitrary domain Ω (not necessarily bounded), using a similar way of proof with estimate (2.1.25) (which does not contain constants dependent on Ω) instead of (6.8.10), and with the space V instead Y, one obtains

$$\begin{aligned}
&\|u'\|_{L_{4/3}(t,T;V^*)} + \|\tau'\|_{L_2(t,T;H^{-2})} \\
&\quad \le K_5\big(\|u\|_{L_\infty(t,T;H)}, \|u\|_{L_2(t,T;V)}, \|\tau\|_{L_\infty(t,T;L_2)}, \\
&\qquad \|f\|_{L_2(t,T;V^*)}, T - t\big)
\end{aligned} \tag{6.8.11}$$

where K_5 is independent of Ω.

6.8.2 Existence and structure of uniform attractors

Let us now construct the minimal uniform trajectory attractors and the uniform global attractors for problem (6.3.1) – (6.3.2). As in Section 6.7, we choose $H \times L_2(\Omega, \mathbb{R}_S^{n\times n})$ as the space E and the space $V_\delta^* \times H^{-\delta}(\Omega, \mathbb{R}_S^{n\times n})$ as the space E_0, where $\delta \in (0, 1]$ is a fixed number.

Fix some $f \in \mathfrak{X}$. We can choose the symbol space Σ, for instance, from the following variants:

a) $\{f\}$;

b) $\Sigma_0 = \{T(t)f \mid t \ge 0\}$;

c) the closure of Σ_0 in the strong topology of $\mathfrak{X}$;

d) the closure of Σ_0 in the weak topology of $\mathcal{X}$;

e) any other subset of $\mathcal{X}$ which contains Σ_0 and satisfies the property

$$\|\sigma\|_{\mathcal{X}} \le \|f\|_{\mathcal{X}} \tag{6.8.12}$$

for all $\sigma \in \Sigma$;

f) any other subset of $\mathcal{X}$ which satisfies (6.8.12) and contains f.

As it was mentioned in Remark 4.3.1, the uniform attractors depend on the choice of the symbol space.

Remark 6.8.2. The sets b) – e) contain Σ_0 (cf. Remark 4.3.3).

Fix a symbol space Σ, given by one of the statements a) – f). Define the trajectory space $\mathcal{H}_\sigma^+$ for a symbol $\sigma \in \Sigma$ as the set of pairs of functions (u, τ) which

i) belong to class (6.7.9);

ii) satisfy identity (6.3.1) and the identity

$$\frac{d}{dt}(u(t), \varphi) - \sum_{i=1}^{n} \Big(u_i(t)u(t), \frac{\partial \varphi}{\partial x_i}\Big) + \mu_1(\nabla u(t), \nabla\varphi) + (\tau(t), \nabla\varphi) = \langle \sigma(t), \varphi\rangle \tag{6.8.13}$$

almost everywhere on $(0, +\infty)$ for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$;

iii) satisfy the energy inequality:

$$\begin{aligned} &\frac{1}{4}\|u\|^2_{L_\infty(t,t+1;H)} + \frac{1}{8\mu_2}\|\tau\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\|u\|^2_{L_2(t,t+1;V)} \\ &\qquad \le e^{-2\gamma t}\Big(\|u\|^2_{L_\infty(0,+\infty;H)} + \frac{1}{2\mu_2}\|\tau\|^2_{L_\infty(0,+\infty;L_2)}\Big) \\ &\qquad\qquad + \frac{2e^{4\gamma} + e^{2\gamma} - 1}{2\mu_1(e^{2\gamma} - 1)}\|f\|^2_{\mathcal{X}} \end{aligned} \tag{6.8.14}$$

for all $t \ge 0$ where γ is as in Theorem 6.7.1.

Remark 6.8.3. The same arguments as in Remark 6.7.1 show that

$$\mathcal{H}_\sigma^+ \subset C([0, +\infty); E_0) \cap L_\infty(0, +\infty; E)$$

for all $\sigma \in \Sigma$.

Remark 6.8.4. It is clear that on account of inequality (6.8.14) the family of trajectory spaces $\mathcal{H}_\sigma^+$, $\sigma \in \Sigma$ is not translation-coordinated (cf. Remark 4.3.2).

Theorem 6.8.2 (cf. [77]). *Given $a \in H$, $\tau_0 \in L_2(\Omega, \mathbb{R}_S^{n\times n})$, $\sigma \in \Sigma$, there is a pair of functions (a trajectory) $(u, \tau) \in \mathcal{H}_\sigma^+$ satisfying initial condition (6.3.5) .*

Proof. Take an increasing sequence of positive numbers $T_m \to \infty$. Put in Theorems 6.5.1 and 6.8.1 σ instead of f. Then, by Theorem 6.5.1, for every natural m there is a pair (u_m, τ_m) which belongs to class (6.5.1) with $T = T_m$, and satisfies identities (6.3.1), (6.3.2) for almost everywhere $t \in (0, T_m)$ and for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$. By Theorem 6.8.1, all pairs (u_m, τ_m) satisfy inequality (6.8.1) for $t \in [0, T_m - 1]$ with σ instead of f. Since $\|\sigma\|_{\mathfrak{X}} \leq \|f\|_{\mathfrak{X}}$, they satisfy it also with f. Denote by $\widetilde{u}_m$ and $\widetilde{\tau}_m$ the functions which are equal to u_m and τ_m in $[0, T_m]$ and vanish on $(T_m, +\infty)$. Then, without loss of generality one may assume that there exist limits

$$u = \lim_{m\to\infty} \widetilde{u}_m, \text{which is } *\text{-weak in } L_\infty(0, +\infty; H);$$

$$\tau = \lim_{m\to\infty} \widetilde{\tau}_m, \text{which is } *\text{-weak in } L_\infty(0, +\infty; L_2).$$

Fix an arbitrary interval $[0, T]$. Then estimate (6.8.1) for all pairs (u_m, τ_m) yields uniform boundedness of the sequence $\{\widetilde{u}_m\}$ in $L_2(0, T; V)$. Therefore without loss of generality we may consider that $\widetilde{u}_m \rightharpoonup u$ weakly in $L_2(0, T; V)$. It is easy to see that the pair (u, τ) satisfies inequality (6.8.1) (and, hence, (6.8.14)) for $t \in [0, T - 1]$, and that, for $T_m \geq T$, the functions $\widetilde{u}_m$ and $\widetilde{\tau}_m$ coincide with u_m and τ_m on the segment $[0, T]$.

Just as in the proof of Theorems 6.4.1, 6.5.1, 6.7.2, estimate (6.8.1) for all pairs (u_m, τ_m) and Lemma 6.8.1 ensure convergence of all terms in identities (6.3.1), (6.8.13) with (u_m, τ_m) substituted there, $T_m \geq T$, to the corresponding terms in (6.3.1), (6.8.13) with (u, τ) in the sense of scalar distributions on $(0, T)$. Therefore the pair (u, τ) satisfies identities (6.3.1), (6.8.13) on $(0, T)$ for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$, belongs to class (6.7.8) and satisfies condition (6.3.5). Since T was arbitrary, the pair (u, τ) satisfies conditions i) – iii) of the definition of $\mathcal{H}_\sigma^+$. □

The main results of this section are

Theorem 6.8.3 (cf. [77]). *There exists a minimal uniform (with respect to $\sigma \in \Sigma$) trajectory attractor $\mathfrak{U}_J$ for problem (6.3.1) – (6.3.2).*

Theorem 6.8.4 (cf. [77]). *In the space $H \times L_2$ there is a uniform (with respect to $\sigma \in \Sigma$) global attractor $\mathcal{A}_J$ (in $V_\delta^* \times H^{-\delta}$) for problem (6.3.1) – (6.3.2).*

Proof. Consider the set P which consists of pairs

$$(u, \tau) \in C([0, +\infty); E_0) \cap L_\infty(0, +\infty; E)$$

satisfying inequalities

$$
\begin{aligned}
&\frac{1}{4}\|u\|^2_{L_\infty(t,t+1;H)} + \frac{1}{8\mu_2}\|\tau\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\|u\|^2_{L_2(t,t+1;V)} \\
&\qquad \le \frac{2e^{4\gamma} + e^{2\gamma} - 1}{\mu_1(e^{2\gamma} - 1)}\|f\|^2_{\mathfrak{X}}, \\
&\|u'\|_{L_{4/3}(t,t+1;Y^*)} + \|\tau'\|_{L_2(t,t+1;H^{-2})} \\
&\qquad \le K_1\big(\|u\|_{L_\infty(t,t+1;H)}, \|u\|_{L_2(t,t+1;V)}, \|\tau\|_{L_\infty(t,t+1;L_2)}, \|f\|_{\mathfrak{X}}, 1, \Omega\big)
\end{aligned}
$$

for all $t \ge 0$, where K_1 is the constant from estimate (6.8.9). Just as in the proof of Theorem 6.7.3, one shows that $\overline{P}$ (its closure in $C([0, +\infty); E_0)$) is a uniform trajectory semiattractor.

Now from Theorem 4.3.1 we get Theorem 6.8.3, and Theorems 4.3.3 and 6.8.3 imply Theorem 6.8.4. □

6.9 Stationary boundary-value problem for the Jeffreys model

6.9.1 Strong and weak statements of the stationary problem

One of the interesting and important problems in the theory of evolutionary equations, and of the equations from fluid mechanics, in particular, is the study of *stationary*, i.e. time-independent, solutions. Such solutions correspond to the *steady* flow regime.

The boundary value problem describing the steady flow of the Jeffreys medium with constitutive equation (1.3.12) easily results from problem (6.2.1) – (6.2.4):

$$\sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} + \operatorname{grad} p = \operatorname{Div} \sigma + f, \tag{6.9.1}$$

$$\sigma + \lambda_1 \sum_{i=1}^{n} u_i \frac{\partial \sigma}{\partial x_i} = 2\eta\Big(\mathcal{E} + \lambda_2 \sum_{i=1}^{n} u_i \frac{\partial \mathcal{E}}{\partial x_i}\Big), \tag{6.9.2}$$

$$\operatorname{div} u = 0, \tag{6.9.3}$$

$$u\big|_{\partial\Omega} = 0. \tag{6.9.4}$$

Note that Ω here is an arbitrary (possibly unbounded) domain in $\mathbb{R}^n$, $n = 2, 3$.

Observe that stationary solutions may exist only for the autonomous problems (when f depends only on x). Let $f \in Y^*(\Omega)$.

Definition 6.9.1. A *weak solution* to problem (6.9.1) – (6.9.4) is a pair of functions $u \in Y,\ \sigma \in L_2(\Omega, \mathbb{R}_S^{n\times n})$ such that

$$(\sigma, \nabla\varphi) - \sum_{i=1}^{n} \big(u_i u, \frac{\partial\varphi}{\partial x_i}\big) = \langle f, \varphi\rangle_{Y^*\times Y}, \tag{6.9.5}$$

$$(\sigma, \Phi) - \lambda_1 \sum_{i=1}^{n} (u_i \sigma, \frac{\partial\Phi}{\partial x_i}) = -2\eta(u, \operatorname{Div}\Phi) - 2\eta\lambda_2 \sum_{i=1}^{n} \big(u_i \mathcal{E}(u), \frac{\partial\Phi}{\partial x_i}\big) \tag{6.9.6}$$

for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$.

This weak setting can by realized in the framework of the general scheme of Section 6.1.1 just as it was done for problem (6.2.1) – (6.2.5).

Theorem 6.9.1 (see [68]). *Let $f \in Y^*$. Then there exists a weak solution to problem (6.9.1) – (6.9.4).*

6.9.2 Auxiliary problem and a priori bound

As in the non-stationary case, we study an auxiliary problem first.

Using the notations $\mu_1 = \eta\frac{\lambda_2}{\lambda_1},\ \mu_2 = \frac{\eta-\mu_1}{\lambda_1},\ \tau = \sigma - 2\mu_1\mathcal{E}$, as in Section 6.3, we rewrite system (6.9.5) – (6.9.6) in the form:

$$\frac{1}{\lambda_1}(\tau, \Phi) - \sum_{i=1}^{n} \big(u_i \tau, \frac{\partial\Phi}{\partial x_i}\big) + 2\mu_2(u, \operatorname{Div}\Phi) = 0, \tag{6.9.7}$$

$$-\sum_{i=1}^{n} \big(u_i u, \frac{\partial\varphi}{\partial x_i}\big) + \mu_1(\nabla u, \nabla\varphi) + (\tau, \nabla\varphi) = \langle f, \varphi\rangle. \tag{6.9.8}$$

Now, consider the following auxiliary problem:

$$\frac{1}{\lambda_1}(\tau, \Phi) - \delta \sum_{i=1}^{n} \big(u_i \tau, \frac{\partial\Phi}{\partial x_i}\big) + 2\mu_2\delta(u, \operatorname{Div}\Phi) + \varepsilon(\nabla\tau, \nabla\Phi) = 0, \tag{6.9.9}$$

$$-\delta \sum_{i=1}^{n} \big(u_i u, \frac{\partial\varphi}{\partial x_i}\big) + \mu_1(\nabla u, \nabla\varphi) + \delta(\tau, \nabla\varphi) = \delta\langle f, \varphi\rangle \tag{6.9.10}$$

for all $\varphi \in Y,\ \Phi \in H_0^1$ (here $\varepsilon > 0$ and $0 \le \delta \le 1$ are parameters).

Let us prove the following a priori estimate:

Lemma 6.9.1. *Let Ω be bounded, and let a pair $(u \in Y, \tau \in H_0^1(\Omega, \mathbb{R}_S^{n\times n}))$ be a solution to (6.9.9), (6.9.10). Then the following a priori estimate takes place.*

$$\mu_1\|u\|_Y^2 + \frac{1}{2\lambda_1\mu_2}\|\tau\|^2 + \frac{\varepsilon}{2\mu_2}\|\tau\|_Y^2 \le \frac{1}{\mu_1}\|f\|_{Y^*}^2. \tag{6.9.11}$$

Proof. Put $\varphi = u$ in (6.9.10), and $\Phi = \dfrac{\tau}{2\mu_2}$ in (6.9.9), and add the obtained equalities. Taking into account (6.1.18) – (6.1.20), we get:

$$\mu_1(\nabla u, \nabla u) + \frac{1}{2\lambda_1\mu_2}(\tau, \tau) + \frac{\varepsilon}{2\mu_2}(\nabla\tau, \nabla\tau) = \delta\langle f, u\rangle. \tag{6.9.12}$$

Since

$$\delta\langle f, u\rangle \le \|f\|_{Y^*}\|u\|_Y, \tag{6.9.13}$$

(6.9.12) yields

$$\mu_1\|u\|_Y^2 \le \|f\|_{Y^*}\ \|u\|_Y,$$

so

$$\|u\|_Y \le \frac{1}{\mu_1}\|f\|_{Y^*}. \tag{6.9.14}$$

Now (6.9.12), (6.9.13) and (6.9.14) imply (6.9.11). □

6.9.3 Solvability of the auxiliary problem

Theorem 6.9.2. *Let Ω be bounded and $f \in Y^*$. Then there is a solution ($u \in Y$, $\tau \in H_0^1(\Omega, \mathbb{R}_S^{n\times n})$) to problem (6.9.9), (6.9.10).*

Proof. Let us introduce auxiliary operators by the following formulas (in these formulas φ and Φ are arbitrary elements of Y and $H_0^1(\Omega, \mathbb{R}_S^{n\times n})$, respectively):

$$K : Y \to Y^*, \quad \langle K(u), \varphi\rangle = -\sum_{i=1}^{n}\Big(u_i u, \frac{\partial\varphi}{\partial x_i}\Big),$$

$$A : Y \to Y^*, \quad \langle A(u), \varphi\rangle = \mu_1(\nabla u, \nabla\varphi),$$

$$A_\varepsilon : H_0^1 \to H^{-1}, \quad \langle A_\varepsilon(\tau), \Phi\rangle = \varepsilon(\nabla\tau, \nabla\Phi) + \frac{1}{\lambda_1}(\tau, \Phi),$$

$$\tilde{A} : Y \times H_0^1 \to Y^* \times H^{-1}, \quad \tilde{A}(u, \tau) = (A(u), A_\varepsilon(\tau)),$$

$$N_1 : H_0^1 \to Y^*, \quad \langle N_1(\tau), \varphi\rangle = (\tau, \nabla\varphi),$$

$$N_2 : Y \to H^{-1}, \quad \langle N_2(u), \Phi\rangle = 2\mu_2(u, \operatorname{Div}\Phi),$$

$$\tilde{K} : Y \times H_0^1 \to H^{-1}, \quad \langle \tilde{K}(u, \tau), \Phi)\rangle = -\sum_{i=1}^{n}(u_i\tau, \frac{\partial\Phi}{\partial x_i}),$$

$$Q : Y \times H_0^1 \to Y^* \times H^{-1}, \quad Q(u, \tau) = (K(u) + N_1(\tau) - f,\ \tilde{K}(u, \tau) + N_2(u)).$$

Then system (6.9.9), (6.9.10) is equivalent to the operator equation

$$\tilde{A}(u, \tau) + \delta Q(u, \tau) = 0. \tag{6.9.15}$$

The linear operator N_1 is bounded as a map from L_2 into Y^*. But the embedding of H_0^1 into L_2 is compact, so the operator N_1 is compact as an operator from H_0^1 into Y^*. Similarly, the operator N_2 is compact since $Y \cong V$ (Ω is bounded now) is imbedded into L_2 compactly.

By Hölder's inequality (2.1.1), the operator K is continuous as map from L_4 into Y^*. But $Y \subset L_4$ compactly, so the operator K is compact (as map from Y into Y^*). Similarly, the operator $\tilde{K}$ is continuous as map from $L_4 \times L_4$ into H^{-1}, and the operator $\tilde{K} : Y \times H_0^1 \to H^{-1}$ is compact.

Hence, the operator Q is compact.

By Theorem 3.1.1 the operator $\tilde{A}$ is invertible. Rewrite equation (6.9.15) in the form

$$(u,\tau) + \delta\tilde{A}^{-1}Q(u,\tau) = 0. \tag{6.9.16}$$

Due to Lemma 6.9.1, equation (6.9.16) has no solutions on the boundary of a sufficiently large ball B in $Y \times H_0^1$ independent of δ. Then we can use the Leray–Schauder degree $\deg_{LS}(I+\delta\tilde{A}Q, B, 0)$ (see Section 3.2.2). By the homotopic invariance property of the degree,

$$\deg_{LS}(I+\delta\tilde{A}Q, B, 0) = \deg_{LS}(I, B, 0) = 1.$$

Hence, equation (6.9.16), and, therefore, system (6.9.9), (6.9.10), has a solution in B at every $\delta \in [0,1]$. □

6.9.4 Proof of Theorem 6.9.1.

Denote by Ω_m the intersection of Ω with the ball B_m of radius m centered at the origin in the space $\mathbb{R}^n, m = 1, 2, \ldots$.

Consider the "restriction" of f on Ω_m, $f|_{\Omega_m} \in Y^*(\Omega_m)$, which is defined by the formula

$$\langle f|_{\Omega_m}, \varphi\rangle = \langle f, \tilde{\varphi}\rangle$$

where φ is a function from $Y(\Omega_m)$, and $\tilde{\varphi}$ coincides with φ in Ω_m, and is identically zero in $\Omega\backslash\Omega_m$. Obviously, $\|f|_{\Omega_m}\|_{Y^*(\Omega_m)} \le \|f\|_{Y^*(\Omega)}$.

On each Ω_m, consider problem (6.9.9), (6.9.10) with $f = f|_{\Omega_m}$, $\varepsilon = \frac{1}{m}$, $\delta = 1$. By Theorem 6.9.2 these problems have at least one solution (u_m, τ_m). Denote by $(\tilde{u}_m, \tilde{\tau}_m)$ the functions which coincide with u_m and τ_m, respectively, in Ω_m, and are identically zero in $\Omega\backslash\Omega_m$. By Lemma 6.9.1, the norms $\|\tilde{u}_m\|_{Y(\Omega)} = \|u_m\|_{Y(\Omega_m)}$ and $\|\tilde{\tau}_m\|_{L_2(\Omega)} = \|\tau_m\|_{L_2(\Omega_m)}$ are uniformly bounded. Therefore, as $m \to \infty$, without loss of generality we may assume that $\tilde{u}_m \to \tilde{u}_0$ weakly in Y, $\tilde{\tau}_m \to \tilde{\tau}_0$ weakly in L_2. Let us show that $(\tilde{u}_0, \ \tilde{\tau}_0)$ is a solution to problem (6.9.7), (6.9.8).

Take arbitrary $\varphi \in \mathcal{V}$, $\Phi \in C_0^\infty$. At some k the supports of φ and Φ are contained in Ω_k. Denote by u_m^* the functions which coincide with $\tilde{u}_m$ in $\Omega \bigcap B_k$, and are identically zero in $B_k\backslash\Omega$. It is clear that $u_m^* \to u_0^*$ weakly in $W_2^1(B_k)$, that is, strongly in $L_4(B_k)$.

Therefore, all terms from (6.9.9), (6.9.10) with $\varepsilon = \frac{1}{m}$, $\delta = 1$, $u = u_m$, $\tau = \tau_m$ converge to the corresponding terms in (6.9.7), (6.9.8), and

$$|\varepsilon(\nabla\tau_m, \nabla\Phi)| = \Big|\frac{1}{m}(\tilde{\tau}_m, \Delta\Phi)\Big| \le \frac{1}{m}\|\tilde{\tau}_m\|\,\|\Delta\Phi\| \to 0.$$

Thus, the pair $(\tilde{u}_0, \tilde{\tau}_0)$ satisfies identities (6.9.7), (6.9.8) for all $\varphi \in \mathcal{V}$, $\Phi \in C_0^\infty$. Let $\tilde{\sigma}_0 = \tilde{\tau}_0 + 2\mu_1\mathcal{E}(\tilde{u}_0)$. It is clear that $\tilde{\sigma}_0 \in L_2$. Then $(\tilde{u}_0, \tilde{\sigma}_0)$ is a solution to problem (6.9.5), (6.9.6), i.e. a weak solution to problem (6.9.1) – (6.9.4).

Chapter 7

The regularized Jeffreys model

7.1 Formulation of the problem and the main results

Consider the motion of homogeneous incompressible Jeffreys' viscoelastic medium filling a domain Ω in $\mathbb{R}^n$, $2 \le n \le 4$, on a time interval $(0,T)$, $T > 0$. We recall (Section 1.3.3) that expressing σ from the constitutive relation (1.3.12) and substituting it in equation (1.1.12) one gets (1.3.14). In this chapter we are going to investigate the equation of motion (1.3.14) instead of system (6.2.1), (6.2.2). Let us point out that equation (1.3.14) seems to be more convenient for the applications and numerical schemes, since (in the three-dimensional case) it contains only four scalar function variables v_1, v_2, v_3, p, whereas system (6.2.1), (6.2.2) has three velocity variables, one pressure variable, and six stress variables $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}$.

Hereafter we assume that Ω is a sufficiently regular bounded domain, and use Einstein's summation agreement.

Equation (1.3.14) contains the function z which describes the trajectories of particles of a medium. More precisely, $z(\varsigma, t, x)$ shows which position, at the moment ς, is occupied with the particle which at the moment t is at the point x (in our case, $x \in \Omega$). Rewrite the properties (1.3.4), (1.3.5) of z in the equivalent form

$$z(\varsigma, t, x) = x + \int_t^{\varsigma} v(s, z(s,t,x))\, ds. \tag{7.1.1}$$

Then the trajectories of the particles of the medium are determined by the velocity field v as the solution of this integral equation.

Combining (1.3.14) and (7.1.1) with the incompressibility condition (1.1.10), the no-slip condition (1.1.15), and the initial condition, we arrive at the following problem

$$\begin{aligned} &\frac{\partial v}{\partial t} + v_i \frac{\partial v}{\partial x_i} - \mu \operatorname{Div} \int_0^t e^{-\frac{t-s}{\lambda}} \mathcal{E}(v)(s, z(s,t,x))\, ds - \frac{\mu_0}{2} \Delta v \\ &\quad = -\operatorname{grad} p + f, \quad (t,x) \in (0,T) \times \Omega, \end{aligned} \tag{7.1.2}$$

$$z(\varsigma, t, x) = x + \int_t^{\varsigma} v(s, z(s,t,x))\, ds, \quad \varsigma \in [0,T], \ \ (t,x) \in (0,T) \times \Omega, \tag{7.1.3}$$

$$\operatorname{div} v = 0, \quad (t,x) \in (0,T) \times \Omega, \tag{7.1.4}$$

$$v\,|_{(0,T)\times\partial\Omega} = 0, \tag{7.1.5}$$

$$v(0,x) = a(x), \quad x \in \Omega. \tag{7.1.6}$$

Here we assumed for simplicity that

$$\sigma_0 - 2\eta\frac{\lambda_2}{\lambda_1}\mathcal{E}(a) = 0 \tag{7.1.7}$$

and used the notations

$$\mu_0 = 2\eta\frac{\lambda_2}{\lambda_1} > 0, \ \mu = 2\eta\frac{\lambda_1 - \lambda_2}{\lambda_1^2} > 0. \tag{7.1.8}$$

Let us introduce some additional notations required in the sequel.

First we point out that in the space $V(\Omega)$ we shall sometimes use the scalar product $(u,v)_V = \int_\Omega \mathcal{E}_{ij}(u) \cdot \mathcal{E}_{ij}(v)\,dx = \frac{1}{2}(u,v)_Y$ and the corresponding Euclid norm $\|v\|_V$.

We shall use the following notations for the vector function spaces:

$E = L_2(0,T;V)$	with the norm $\|v\|_E = \|v\|_{L_2(0,T;V)}$	for $v \in E$;
$E^* = L_2(0,T;V^*)$	with the norm $\|f\|_{E^*} = \|f\|_{L_2(0,T;V^*)}$	for $f \in E^*$;
$E_1^* = L_1(0,T;V^*)$	with the norm $\|f\|_{E_1^*} = \|f\|_{L_1(0,T;V^*)}$	for $f \in E_1^*$.

Denote by $C^1D(\overline{\Omega})$ the set of one-to-one maps $z : \overline{\Omega} \to \overline{\Omega}$ coinciding with the identity map on $\partial\Omega$ and having continuous partial derivatives of the first order such that $\det\left(\frac{\partial z}{\partial x}\right) = 1$ at each point of $\overline{\Omega}$. We shall use the norm of the space of continuous functions $C(\overline{\Omega})^n$ in this set.

Let us introduce the set $CG = C([0,T] \times [0,T], C^1D(\overline{\Omega}))$. Note that $CG \subset C([0,T] \times [0,T], C^1(\overline{\Omega})^n)$, and CG will be considered as a metric space with the metric of space $C([0,T] \times [0,T], C(\overline{\Omega})^n)$.

Let $Q_T = (0,T) \times \Omega$.

Equation (7.1.2) includes an integral along the trajectories of motion of particles of the medium (cf. Section 1.3.3). Therefore the trajectories should be unambiguously determined by the field of velocities $v(t,x)$. In other words, equation (7.1.3) should have a unique solution for a given velocity field $v(t,x)$. However, existence of solutions for equation (7.1.3) at fixed v is known only in the case $v \in L_1(0,T;C(\overline{\Omega})^n)$, and this solution is unique for $v \in L_1(0,T;C^1(\overline{\Omega})^n)$ such that $v\,|_{(0,T)\times\partial\Omega} = 0$ (see, for example, [46, 47], [48, Lemmas 1 and 2]). Therefore, for weak solutions, the motion trajectories cannot be determined unambiguously. A possible way out of the situation (see e.g. [39]) is a smoothing (a regularization) of the velocity field and determination of the trajectories $z = Z_\delta(v)$ for the smoothed velocity field $S_\delta(v)$.

Let us choose some linear regularization operator $S_\delta : H \to C^1(\overline{\Omega})^n \cap V$ for $\delta > 0$ such that the map $S_\delta : L_2(0,T;H) \to L_2(0,T;C^1(\overline{\Omega})^n \cap V)$ generated by the operator is continuous and the operators $S_\delta : L_2(0,T;H) \to L_2(0,T;H)$ converge strongly to the identity operator I as $\delta \to 0$. A construction of such an operator is given in Section 7.7.

Replace equation (7.1.3) in system (7.1.2) – (7.1.6) with the following equation:

$$z(\varsigma,t,x) = x + \int_t^\varsigma S_\delta v(s,z(s,t,x))\,ds, \quad \varsigma,t \in (0,T), \ \ x \in \Omega. \tag{7.1.9}$$

For each $v \in L_2(0,T;V)$, this equation has a unique solution $z = Z_\delta(v)$ in the class CG. Substituting $Z_\delta(v)$ for z in equation (7.1.2), we arrive at the regularized problem

$$\begin{aligned}&\frac{\partial v}{\partial t} + v_i\frac{\partial v}{\partial x_i} - \mu\,\mathrm{Div}\int_0^t e^{-\frac{t-s}{\lambda}}\mathcal{E}(v)(s,Z_\delta(v)(s,t,x))\,ds - \frac{\mu_0}{2}\Delta v\\ &\quad = -\mathrm{grad}\,p + f, \quad (t,x) \in (0,T)\times\Omega,\end{aligned} \tag{7.1.10}$$

$$\mathrm{div}\ v = 0, \quad (t,x) \in (0,T)\times\Omega, \tag{7.1.11}$$

$$v\,|_{(0,T)\times\partial\Omega} = 0, \tag{7.1.12}$$

$$v(0,x) = a(x), \quad x \in \Omega. \tag{7.1.13}$$

In order to define weak solutions for this problem, let us introduce the maps

$$A : V \to V^*,\ \langle A(u),h\rangle = \mu_0(\mathcal{E}(u),\mathcal{E}(h)),\ u,h \in V,$$

$$K : V \to V^*,\ \langle K(u),h\rangle = (u_iu_j,\frac{\partial h_i}{\partial x_j}),\ u,h \in V,$$

$$\langle C(v,z)(t),h\rangle = \mu\left(\int_0^t e^{-\frac{t-s}{\lambda}}\mathcal{E}(v)(s,z(s,t,x))\,ds,\mathcal{E}(h)\right).$$

Here $v \in E$, $z \in CG$, $t \in (0,T)$, $C(v,z)(t) \in V^*$, $h \in V$. Below we show that $C : E \times CG \to E^*$.

Definition 7.1.1. Given $f \in L_1(0,T;V^*)$, $a \in H$, a *weak solution for the regularized problem* (7.1.10) – (7.1.13) is a function $v \in L_2(0,T;V) \cap C_w([0,T];H)$ with $v' \in L_1(0,T;V^*)$ which satisfies equalities

$$v' + A(v) - K(v) + C(v,Z_\delta(v)) = f, \tag{7.1.14}$$

$$v(0) = a. \tag{7.1.15}$$

We point out that a weak solution v belongs to the space

$$W_1 = \{v :\ v \in E,\ v' \in E_1^*\}$$

with the norm $\|v\|_{W_1} = \|v\|_E + \|v'\|_{E_1^*}$. By Lemma 2.2.3, $W_1 \subset C([0,T], V^*)$, so condition (7.1.15) makes sense.

As in the case of problems (6.1.5) – (6.1.8) and (6.2.1) – (6.2.5), using integration by parts one can check that this weak setting can by obtained in the framework of the general scheme of Section 6.1.1.

Note that if v_δ is a strong solution of the regularized problem, then passage to the limit in equality (7.1.9) (and then in (7.1.10)) as $\delta \to 0$ leads us to a solution of problem (7.1.2) – (7.1.6). In Section 7.6, we shall study this question for weak solutions as well.

The solvability of the Cauchy problem (7.1.14) – (7.1.15) will be established provided $f \in L_1(0,T;H^*)+L_2(0,T;V^*)$, i.e. $f = f_1+f_2$ where $f_1 \in L_1(0,T;H^*)$ and $f_2 \in L_2(0,T;V^*)$.

Let us construct the approximating equations. For this purpose, we make modifications in equation (7.1.14), so that all the terms belong to $L_2(0,T;V^*)$. Let $\varepsilon > 0$. Consider the operator

$$K_\varepsilon : V \to V^*,\ \langle K_\varepsilon(u), h\rangle = \left(\frac{u_i u_j}{1+\varepsilon|u|^2}, \frac{\partial h_i}{\partial x_j}\right),$$

and approximate the function f_1 from $L_1(0,T;H^*)$ by functions $f_{1,\varepsilon}$ from $L_2(0,T;H^*)$ so that

$$f_{1,\varepsilon} \to f_1 \ \text{ in } \ L_1(0,T;H^*) \ \text{ as } \ \varepsilon \to 0. \tag{7.1.16}$$

Let f_ε denote the function $f_\varepsilon = f_{1,\varepsilon} + f_2$.

Consider the following Cauchy problem

$$v' + A(v) - K_\varepsilon(v) + C(v, Z_\delta(v)) = f_\varepsilon, \tag{7.1.17}$$

$$v(0) = a, \tag{7.1.18}$$

in the space $W = \{v : v \in E,\ v' \in E^*\}$ with the norm $\|v\|_W = \|v\|_E + \|v'\|_{E^*}$. The space W is a Banach space. Since $W \subset C([0,T], H)$ (due to Lemma 2.2.7), (7.1.18) makes sense.

Introduce the maps $L, G, \mathcal{K}_\varepsilon : W \to E^* \times H$ with the help of the following equalities:

$$L(v) = (v' + A(v), v|_{t=0}),\ \ G(v) = (C(v, Z_\delta(v)), 0),\ \ \mathcal{K}_\varepsilon = (K_\varepsilon, 0).$$

Then problem (7.1.17) – (7.1.18) is equivalent to the operator equation

$$L(v) = \mathcal{K}_\varepsilon - G(v) + (f_\varepsilon, a). \tag{7.1.19}$$

Now we can state the results on existence of weak solutions (see [88]).

Theorem 7.1.1. *Given* $\varepsilon > 0,\ f_\varepsilon \in E^*,\ a \in H$, *problem (7.1.17), (7.1.18) possesses at least one solution* $v \in W$.

Theorem 7.1.2. *Given* $f \in L_1(0,T;H^*) + L_2(0,T;V^*)$ *and* $a \in H$, *problem (7.1.14), (7.1.15) has a solution in* $W_1 \cap C_w([0,T];H)$.

The proof of these statements is given in the following three sections.

7.2 Properties of the operators

In this section we study properties of the operators which appear in equations (7.1.17), (7.1.18), (7.1.19).

Lemma 7.2.1. a) *For* $v \in E$ *one has* $A(v) \in E^*$, *the map* $A : E \to E^*$ *is continuous, and the following estimate is valid*

$$\|A(v)\|_{E^*} \le C_0(1 + \|v\|_E). \tag{7.2.1}$$

b) *For* $v \in E$ *one has* $K(v) \in L_1(0,T;V^*)$ *and* $K_\varepsilon(v) \in L_\infty(0,T;V^*)$. *The maps* $K : E \to L_1(0,T;V^*),\ \ K_\varepsilon : E \to E^*$ *are continuous, and*

$$\|K_\varepsilon(v)\|_{E^*} \le \frac{C_1}{\varepsilon}, \quad \|K_\varepsilon(v)\|_{L_1(0,T;V^*)} \le C_1\|v\|_E^2, \tag{7.2.2}$$

where the constants C_0, C_1 *are independent of* v. *The second estimate holds also for* $\varepsilon = 0$ *(i.e. for* $K_0 = K$*). The map* $K_\varepsilon : W \to E^*$ *is compact for* $\varepsilon > 0$.

These facts can be found in [16, Lemma 2.1 and Theorem 2.2] (cf. also the proof of Theorem 6.3.1, Section 6.3 of this book, and [88]).

As it was mentioned above, $W \subset E \cap C([0,T],H)$. For functions $v \in E \cap C([0,T],H)$, consider the norm

$$\|v\|_{EC} = \max_{0\le t\le T} \|v(t)\| + \|v\|_E.$$

We shall also require the equivalent norms $\|v\|_{k,EC} = \|\bar v\|_{EC}$ where $\bar v(t) = \exp(-kt)\cdot v(t)$, $k \ge 0$.

Similarly, define equivalent norms $\|\cdot\|_{k,E},\ \|\cdot\|_{k,E^*\times H},\ \|\cdot\|_{k,L_2(Q_T)}$ in the spaces $E,\ E^*\times H,\ L_2(Q_T)$.

Theorem 7.2.1. *The map* $L : W \to E^* \times H$ *is invertible, and for all functions* $u, v \in W$ *the estimate*

$$\|v - u\|_{k,EC} \le C_2\|L(v) - L(u)\|_{k,E^*\times H} \tag{7.2.3}$$

holds for any $k \ge 0$. *Here the constant* C_2 *is independent of* u, v, k.

This statement is a particular case of Theorem 2.1 from [16], but as a matter of fact invertibility of L follows easily from Lemma 3.1.3.

Lemma 7.2.2. *For any $v \in E$, $z \in CG$, one has $C(v,z) \in E^*$, and the map*

$$C : E \times CG \to E^*$$

is continuous and bounded.

Proof. By definition of the map C, for $v \in E$, $z \in CG$, $h \in E$ and $t \in (0,T)$:

$$\langle C(v(t), z(\cdot,t,\cdot)), h(t)\rangle = \mu\left(\int_0^t e^{-\frac{t-s}{\lambda}}\mathcal{E}(v)(s,z(s,t,x))\,ds, \mathcal{E}(h(t))\right).$$

Let

$$B : (v,z) \mapsto \int_0^t e^{-\frac{t-s}{\lambda}}\mathcal{E}(s,z(s,t,x))\,ds.$$

It suffices to show that

$$B : E \times CG \to L_2([0,T], L_2(\Omega,\mathbb{R}^{n\times n})),$$

as well as continuity and boundedness of B.

But B is a superposition of an integral operator with the map

$$\begin{aligned} &\Phi : E \times CG \to L_2([0,T]\times[0,T], L_2(\Omega,\mathbb{R}^{n\times n})),\\ &\Phi : (v,z) \mapsto \mathcal{E}(v)(s,z(s,t,x)). \end{aligned} \tag{7.2.4}$$

At any fixed $z \in CG$,

$$\begin{aligned} &\int_0^T\int_0^T \|\mathcal{E}(v)(s,z(s,t,x)) - \mathcal{E}(u)(s,z(s,t,x))\|^2_{L_2(\Omega,\mathbb{R}^{n\times n})}\,ds\,dt\\ &\quad= \int_0^T\int_0^T\int_\Omega |\mathcal{E}|^2(v-u)(s,z(s,t,x))\,dx\,ds\,dt\\ &\quad= T\int_0^T\int_\Omega |\mathcal{E}|^2(v-u)(s,z)\,dz\,ds\\ &\quad= T\int_0^T \|\mathcal{E}(v-u)(s,\cdot)\|^2_{L_2(\Omega,\mathbb{R}^{n\times n})}\,ds = T\|v-u\|^2_E. \end{aligned} \tag{7.2.5}$$

Note that the change of variables $x \to z = z(s,t,x)$ at fixed s,t does not change the norm because $\det\left(\frac{\partial z}{\partial x}\right) = 1$ for all $(s,t,x) \in [0,T]\times[0,T]\times\overline{\Omega}$. Therefore the map Φ is continuous in v uniformly with respect to z.

Now, to prove continuity of Φ, it remains to check its continuity in z at a fixed value of v.

Let z_l be an arbitrary sequence from CG, converging to $z_0 \in CG$ in the space $C([0,T]\times[0,T], C(\overline{\Omega})^n)$, and let $\varepsilon > 0$ be an arbitrary number. Since the set of continuous functions $C(\overline{Q_T}, \mathbb{R}^{n\times n})$ is dense in $L_2(Q_T, \mathbb{R}^{n\times n})$, there exists a continuous $\varepsilon/(3\sqrt{T})$ - approximation of the function $\mathcal{E}(v)$ (denote it by ξ), i.e.

$$\|\mathcal{E}(v) - \xi\|_{L_2(Q_T,\mathbb{R}^{n\times n})} < \frac{\varepsilon}{3\sqrt{T}}.$$

Then

$$\begin{aligned}
&\|\mathcal{E}(v)(s, z_l(s,t,x)) - \mathcal{E}(v)(s, z_0(s,t,x))\|_{L_2([0,T]\times[0,T],L_2(\Omega,\mathbb{R}^{n\times n}))} \\
&\quad\le \|\mathcal{E}(v)(s, z_l(s,t,x)) - \xi(s, z_l(s,t,x))\|_{L_2([0,T]\times[0,T],L_2(\Omega,\mathbb{R}^{n\times n}))} \\
&\qquad + \|\xi(s, z_l(s,t,x)) - \xi(s, z_0(s,t,x))\|_{L_2([0,T]\times[0,T],L_2(\Omega,\mathbb{R}^{n\times n}))} \\
&\qquad + \|\mathcal{E}(v)(s, z_0(s,t,x)) - \xi(s, z_0(s,t,x))\|_{L_2([0,T]\times[0,T],L_2(\Omega,\mathbb{R}^{n\times n}))}.
\end{aligned}$$

Due to the choice of ξ, the first and last terms are less than $\varepsilon/3$. The function ξ is uniformly continuous on Q_T, so the operator of superposition $z \mapsto \xi(\cdot, z)$ is continuous from CG to $C([0,T]\times[0,T], C(\Omega,\mathbb{R}^{n\times n}))$. Therefore,

$$\|\xi(s, z_l(s,t,x)) - \xi(s, z_0(s,t,x))\|_{C([0,T]\times[0,T],C(\Omega,\mathbb{R}^{n\times n}))} \to 0 \ \text{ as } \ l \to \infty.$$

Choosing l large enough, $l > l_0$, we obtain

$$\|\xi(s, z_l(s,t,x)) - \xi(s, z_0(s,t,x))\|_{L_2([0,T]\times[0,T],L_2(\Omega,\mathbb{R}^{n\times n}))} < \varepsilon/3.$$

Hence,

$$\|\mathcal{E}(v)(s, z_l(s,t,x)) - \mathcal{E}(v)(s, z_0(s,t,x))\|_{L_2([0,T]\times[0,T],L_2(\Omega,\mathbb{R}^{n\times n}))} < \varepsilon.$$

Thus, the map Φ is continuous in z.

Estimate (7.2.5) with $u = 0$, continuity of Φ, and boundedness of the integral operator yield boundedness and continuity of the map B. □

Lemma 7.2.3. *The map $Z_\delta : L_2(0,T;H) \to CG$ is continuous.*

Proof. The operator S_δ is continuous from $L_2(0,T;H)$ to $L_2(0,T;C^1(\overline{\Omega})^n \cap V)$. The map Z_δ is a superposition of S_δ with the resolving operator for equation (7.1.3), $Z_0(v) = z$. It remains to show that $Z_0 : L_2(0,T;C^1(\Omega)^n \cap V) \to CG$ is continuous.

Choose an arbitrary sequence

$$\{v_l\}, \ \ v_l \in L_2(0,T;C^1(\Omega)^n \cap V), \ \ v_l \to v_0 \ \text{ as } l \to \infty.$$

By [48, Lemmas 1 and 3], for every $v = v_l$, $l = 0,1,2,\dots$, there is a unique solution z_l in CG to equation (7.1.3), and one has the estimate

$$\|z_l(\varsigma,t,\cdot) - z_0(\varsigma,t,\cdot)\|_{C(\overline{\Omega})^n} \le M\left|\int_t^{\varsigma} \|v_l(s,\cdot) - v_0(s,\cdot)\|_{C(\overline{\Omega})^n}\, ds\right|$$

with some constant M independent of l, ς, t. Then

$$\|z_l - z_0\|_{C([0,T]\times[0,T],C(\overline{\Omega})^n)} \le M\|v_l - v_0\|_{L_1(0,T;C^1(\Omega)^n)}$$
$$\le C\|v_l - v_0\|_{L_2(0,T;C^1(\Omega)^n)}.$$

Since $v_l \to v_0$ as $l \to \infty$ in the space $L_2(0,T;C^1(\Omega)^n)$, we get $z_l = Z_0(v_l) \to Z_0(v_0) = z_0$ strongly in the space $C([0,T]\times[0,T],C(\overline{\Omega})^n)$. □

Lemma 7.2.4. *For $z \in CG$, $u, v \in E$ there is the estimate*

$$\|C(v,z) - C(u,z)\|_{k,E^*} \le \mu\sqrt{\frac{T}{2k}}\,\|u - v\|_{k,E}. \tag{7.2.6}$$

Proof. Let $\overline{v}(t) = \exp(-kt)v(t)$, $\overline{u}(t) = \exp(-kt)u(t)$. By the definition of C, for $h \in E$ we have

$$\big\langle \exp(-kt)C(v(t),z(\cdot,t,\cdot)) - \exp(-kt)C(u(t),z(\cdot,t,\cdot)),h(t)\big\rangle_{E^*\times E}$$
$$= \mu\int_0^T\int_\Omega\int_0^t e^{-(t-s)(\frac{1}{\lambda}+k)}\mathcal{E}_{ij}(\overline{v}-\overline{u})(s,z(s,t,x))\,ds\,\mathcal{E}_{ij}(h)(t,x)\,dx\,dt$$

Then with the help of Hölder's inequality we get

$$\Big|\big\langle \exp(-kt)C(v(t),z(\cdot,t,\cdot)) - \exp(-kt)C(u(t),z(\cdot,t,\cdot)),h(t)\big\rangle\Big|$$
$$\le \mu\int_0^T\int_0^t e^{-(t-s)(\frac{1}{\lambda}+k)}\Big(\int_\Omega|\mathcal{E}|^2(\overline{v}-\overline{u})(s,z(s,t,x))\,dx\Big)^{1/2}$$
$$\cdot\Big(\int_\Omega|\mathcal{E}|^2(h)(t,x)\,dx\Big)^{1/2}ds\,dt$$

$$= \mu\int_0^T\int_0^t e^{-(t-s)(\frac{1}{\lambda}+k)}\Big(\int_\Omega|\mathcal{E}|^2(\overline{v}-\overline{u})(s,z)\,dz\Big)^{1/2}\cdot\|h(t,\cdot)\|_V\,ds\,dt$$
$$= \mu\int_0^T\int_0^t e^{-(t-s)(\frac{1}{\lambda}+k)}\|(\overline{v}-\overline{u})(s,\cdot)\|_V\cdot\|h(t,\cdot)\|_V\,ds\,dt$$
$$\le \mu\int_0^T\Big(\int_0^t e^{-2(t-s)(\frac{1}{\lambda}+k)}\,ds\Big)^{1/2}\Big(\int_0^t\|(\overline{v}-\overline{u})(s,\cdot)\|_V^2\,ds\Big)^{1/2}$$
$$\cdot\|h(t,\cdot)\|_V\,dt$$
$$\le \mu\|\overline{v}-\overline{u}\|_E\cdot\|h\|_E\cdot\Big(\int_0^T\int_0^t e^{-2(t-s)(\frac{1}{\lambda}+k)}\,ds\,dt\Big)^{1/2}.$$

Let us estimate the remaining integral. Denote $\lambda_0 = 2(\frac{1}{\lambda}+k)$, then

$$\int_0^T\int_0^t e^{-(t-s)\lambda_0}\,ds\,dt = \frac{1}{\lambda_0}\int_0^T(1-e^{-t\lambda_0})\,dt \le \frac{1}{\lambda_0}\int_0^T dt = \frac{T}{\lambda_0}.$$

Thus,

$$\left|\langle e^{-kt}C(v,z) - e^{-kt}C(u,z), h\rangle\right| \le \mu\sqrt{\frac{T\lambda}{2+2\lambda k}}\,\|\overline{v}-\overline{u}\|_E \cdot \|h\|_E,$$

whence

$$\begin{aligned}\|C(v,z)-C(u,z)\|_{k,E^*} &= \|e^{-kt}(C(v,z)-C(u,z))\|_{E^*}\\ &\le \mu\sqrt{\frac{T\lambda}{2+2\lambda k}}\,\|\overline{v}-\overline{u}\|_E \le \mu\sqrt{\frac{T}{2k}}\,\|v-u\|_{k,E}. \quad \square\end{aligned}$$

These lemmas allow us to investigate properties of the map G.

We recall [86] that, for given bounded maps $F, \mathcal{A} : X \to Y$ (X and Y are Banach spaces), the map F is called $\mathcal{A}$*-condensing* with respect to a measure of noncompactness [5] γ in Y if

$$\gamma(F(M)) \le q\gamma(\mathcal{A}(M))$$

for any bounded set M in X with some constant $q < 1$.

Let γ_k denote the *Kuratowski measure of noncompactness* [5] in the space E^* with the norm $\|\cdot\|_{k,E^*}$.

Theorem 7.2.2. *The map $G : W \to E^* \times H$ is L-condensing with respect to γ_k for k large enough.*

Proof. Let $M \subset W$ be an arbitrary bounded set. Then it is relatively compact in $L_2(0,T;H)$ by Theorem 2.2.6. Lemma 7.2.3 implies that the set $Z_\delta(M)$ is relatively compact in CG. Then the set $C(v, Z_\delta(M))$ is relatively compact in E^* for any fixed $v \in W$. Besides, for any $z \in Z_\delta(M)$ the map $C(\cdot, z)$ satisfies the Lipschitz condition (7.2.6) with constant $\mu\sqrt{\frac{T}{2k}}$. Then by Theorem 1.5.7 from [5] the map $C(v, Z_\delta(v))$ (and hence G) is $\mu\sqrt{\frac{T}{2k}}$*-bounded* with respect to the *Hausdorff measure of noncompactness* χ_k, i.e.

$$\chi_k(G(M)) \le \mu\sqrt{\frac{T}{2k}}\,\chi_k(M).$$

Due to [5, Theorem 1.1.7],

$$\chi_k(M_*) \le \gamma_k(M_*) \le 2\chi_k(M_*)$$

for any bounded set M_*. Therefore

$$\gamma_k(G(M)) \le 2\chi_k(G(M)) \le 2\mu\sqrt{\frac{T}{2k}}\,\gamma_k(M). \tag{7.2.7}$$

Estimate (7.2.3) yields

$$\gamma_k(M) \le C_2\, \gamma_k(L(M)). \tag{7.2.8}$$

Estimates (7.2.7) and (7.2.8) imply

$$\gamma_k(G(M)) \le 2C_2\, \mu \sqrt{\frac{T}{2k}}\, \gamma_k(L(M)).$$

Choosing k so that $2C_2\, \mu \sqrt{\frac{T}{2k}} < 1$, we conclude. □

7.3 A priori estimates of solutions and solvability of the approximating equations

Consider the following auxiliary set of operator equations:

$$v' + A(v) - \lambda K_\varepsilon(v) + \lambda C(v, Z_\delta(v)) = f_\varepsilon, \quad \lambda \in [0,1]. \tag{7.3.1$_\lambda$}$$

At $\lambda = 1$ this equation coincides with (7.1.17).

Let us derive some a priori estimates of solutions for this set.

Theorem 7.3.1. *For any solution $v \in W$ to the Cauchy problem (7.3.1$_\lambda$), (7.1.18), $\lambda \in [0,1]$, one has*

$$\|v\|_{EC} \le C\,(1 + \|f_\varepsilon\|_{E^*} + \|a\|), \tag{7.3.2}$$

$$\|v'\|_{E^*} \le C\,(1 + \|f_\varepsilon\|_{E^*} + \|a\|), \tag{7.3.3}$$

where the constant C depends on ε, but is independent of v and $\lambda \in [0,1]$.

Proof. Let $v \in W$ be an arbitrary solution to the Cauchy problem (7.3.1$_\lambda$), (7.1.18) for some $\lambda \in [0,1]$. Then

$$L(v) = \lambda K_\varepsilon - \lambda G(v) + (f_\varepsilon, a). \tag{7.3.4}$$

As $L(0) = 0$, estimate (7.2.3) yields

$$\|v\|_{k,EC} \le C_2 \|L(v)\|_{k,E^*\times H}. \tag{7.3.5}$$

Similarly, $C(0, Z_\delta(v)) = 0$, and estimate (7.2.6) implies the following inequality

$$\|C(v, Z_\delta(v))\|_{k,E^*} = \|C(v, Z_\delta(v)) - C(0, Z_\delta(v))\|_{k,E^*} \le \mu \sqrt{\frac{T}{2k}}\, \|v\|_{k,E}. \tag{7.3.6}$$

Taking into account estimate (7.2.2), we conclude from equality (7.3.4) and estimates (7.3.5), (7.3.6) that

$$\|v\|_{k,EC} \le C\Big(\frac{1}{\varepsilon} + \mu\sqrt{\frac{T}{2k}}\, \|v\|_{k,E} + \|f_\varepsilon\|_{k,E^*} + \|a\|\Big).$$

Since $\|v\|_{k,E} \le \|v\|_{k,EC}$, for k large enough (such that $C\mu\sqrt{\frac{T}{2k}} < \frac{1}{2}$) we obtain

$$\|v\|_{k,EC} \le 2C\left(\frac{1}{\varepsilon} + \|f_\varepsilon\|_{k,E^*} + \|a\|\right).$$

Taking into account the equivalence of norms $\|\cdot\|_{k,EC}$ and $\|\cdot\|_{EC}$, $\|\cdot\|_{k,E^*}$ and $\|\cdot\|_{E^*}$, we get estimate (7.3.2).

Now express v' from equation (7.3.1$_\lambda$):

$$v' = -(A(v) - \lambda K_\varepsilon(v) + \lambda C(v, Z_\delta(v)) - f_\varepsilon).$$

Estimate (7.3.3) follows from estimate (7.3.2) and the properties of the maps A, K_ε, C. □

Let us show the solvability of the Cauchy problem (7.1.17), (7.1.18) via the degree theory for the class of $\mathcal{A}$-condensing perturbations of invertible maps (see [86]).

Proof of Theorem 7.1.1. As it was already mentioned, problem (7.1.17), (7.1.18) is equivalent to the operator equation (7.1.19).

Similarly, the problem (7.3.1$_\lambda$), (7.1.18) is equivalent to the operator equation

$$L(v) - \lambda(K_\varepsilon(v) - G(v)) = (f_\varepsilon, a), \qquad (7.3.7_\lambda)$$

so it suffices to show solvability of (7.3.7$_\lambda$).

Due to Theorem 7.2.2 and Lemma 7.2.1, the map $\lambda(K_\varepsilon - G) : W \times [0,1] \to E^*$ is L-condensing with respect to the Kuratowski measure of noncompactness γ_k. Moreover, a priori estimates (7.3.2), (7.3.3) imply that equations (7.3.7$_\lambda$) have no solutions on the boundary of a ball $B_R \subset W$ of large radius R centered at the origin. Hence, for every $\lambda \in [0,1]$ the topological degree (see [86])

$$\deg_2(L - \lambda(K_\varepsilon - G), \overline{B}_R, (f_\varepsilon, a))$$

is defined. Since the degree of a map is conserved with the change of λ (homotopy invariance), we have

$$\deg_2(L - K_\varepsilon + G, \overline{B}_R, (f_\varepsilon, a)) = \deg_2(L, \overline{B}_R, (f_\varepsilon, a)).$$

The map L is invertible, so the equation

$$L(v) = (f, a)$$

has a unique solution u_0 in W, and for u_0 a priori estimates (7.3.2), (7.3.3) hold. Then $u_0 \in B_R$ and $\deg_2(L, \overline{B}_R, (f_\varepsilon, a)) = 1$ [86]. Therefore

$$\deg_2(L - K_\varepsilon + G, \overline{B}_R, (f_\varepsilon, a)) = 1.$$

Since this degree is nonzero, we have existence of solutions for the operator equation (7.3.7$_\lambda$). □

7.4 A priori estimate and existence of solutions for the regularized problem

In this section we show that the solutions of approximating problems (7.1.17), (7.1.18) converge in some sense to a solution to problem (7.1.14), (7.1.15) as $\varepsilon \to 0$.

For the functions $v \in E \cap C_w([0,T];H)$, consider the norm

$$\|v\|_{EL} = \|v(t)\|_{L_\infty(0,T;H)} + \|v\|_E$$

and the equivalent norms $\|v\|_{k,EL} = \|\bar{v}\|_{EL}$ where $\bar{v}(t) = e^{-kt}v(t)$, $k \geq 0$.

Theorem 7.4.1. *For any solution $v \in W_1 \cap C_w([0,T];H)$ of problem (7.1.17), (7.1.18) with $\varepsilon \geq 0$, one has the estimate*

$$\|v\|_{EL} \leq C\left(1 + \|f_{1,\varepsilon}\|_{L_1(0,T;H^*)} + \|f_2\|_{L_2(0,T;V^*)} + \|a\|\right), \tag{7.4.1}$$

with a constant C independent of ε and δ.

Proof. Let $v \in W_1 \cap C_w([0,T];H)$ be a solution of problem (7.1.17), (7.1.18) for some $\varepsilon \geq 0$. Then

$$v' + A(v) - K_\varepsilon(v) + C(v, Z_\delta(v)) = f_{1,\varepsilon} + f_2. \tag{7.4.2}$$

Let $v(t) = e^{kt}\overline{v}(t)$, $\overline{K}_\varepsilon(\overline{v}) = e^{-kt}K_\varepsilon(e^{kt}\overline{v})$, $\overline{C}(\overline{v}, Z_\delta(e^{kt}\overline{v})) = e^{-kt}C(e^{kt}\overline{v}, Z_\delta(e^{kt}\overline{v}))$, $\overline{f}_{1,\varepsilon} = e^{-kt}f_{1,\varepsilon}$, $\overline{f}_2 = e^{-kt}f_2$. Multiplying (7.4.2) by e^{-kt}, we obtain

$$\overline{v}' + k\overline{v} + A(\overline{v}) - \overline{K}_\varepsilon(\overline{v}) + \overline{C}(\overline{v}, Z_\delta(e^{kt}\overline{v})) = \overline{f}_{1,\varepsilon} + \overline{f}_2. \tag{7.4.3}$$

Consider the values of the functionals in the left-hand and right-hand members of (7.4.3) on the function $\overline{v}$:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\overline{v}(t)\|^2 + k\|\overline{v}(t)\|^2 + (A(\overline{v}(t)), \overline{v}(t)) - (\overline{K}_\varepsilon(\overline{v}(t)), \overline{v}(t))\\ &\qquad = -(\overline{C}(\overline{v}, Z_\delta(e^{kt}\overline{v}))(t), \overline{v}(t)) + (\overline{f}_{1,\varepsilon}, \overline{v}(t)) + (\overline{f}_2, \overline{v}(t)).\end{aligned} \tag{7.4.4}$$

Formula (6.1.21) with $\tau \equiv 0$ yields $(\overline{K}_\varepsilon(\overline{v}(t)), \overline{v}(t)) = 0$ for all $t \in [0,T]$. Therefore, integrating both parts of (7.4.4) along a segment $[0,t]$, we get

$$\begin{aligned}&\frac{1}{2}\|\overline{v}(t)\|^2 + k\|\overline{v}\|^2_{L_2(0,T;H)} + \mu_0\|\overline{v}\|^2_{L_2(0,T;V)}\\ &\quad = \frac{1}{2}\|\overline{a}\|^2 - \int_0^t (\overline{C}(\overline{v}, Z_\delta(e^{k\varsigma}\overline{v}))(\varsigma), \overline{v}(\varsigma))\,d\varsigma + \int_0^t (\overline{f}_{1,\varepsilon}(\varsigma), \overline{v}(\varsigma))\,d\varsigma\\ &\qquad + \int_0^t (\overline{f}_2(\varsigma), \overline{v}(\varsigma))\,d\varsigma.\end{aligned}$$

From here and from estimate (7.2.6) for $u = 0$ with the help of Cauchy's inequality we arrive at

$$
\begin{aligned}
&\frac{1}{2}\,\|\overline{v}(t)\|^2 + k\|\overline{v}\|^2_{L_2(0,T;H)} + \mu_0\|\overline{v}\|^2_{L_2(0,T;V)} \\
&\quad \le \frac{1}{2}\,\|\overline{a}\|^2 - \mu\sqrt{\frac{T}{2k}}\|\overline{v}\|^2_E + \|\overline{f}_{1,\varepsilon}\|_{L_1(0,T;H^*)}\cdot\|\overline{v}\|_{L_\infty(0,T;H)} \\
&\qquad + \|\overline{f}_2\|_{L_2(0,T;V^*)}\cdot\|\overline{v}\|_{L_2(0,T;V)}.
\end{aligned}
$$

Considering k large enough so that $\mu\sqrt{\frac{T}{2k}} < \frac{\mu_0}{2}$, and using Cauchy's inequality, we get the estimate

$$
\begin{aligned}
&\|\overline{v}\|^2_{L_\infty(0,T;H)} + 2k\|\overline{v}\|^2_{L_2(Q_T)} + \mu_0\|\overline{v}\|_E \\
&\quad \le \|a\|^2_H + \frac{1}{2}\,\|\overline{v}\|^2_{L_\infty(0,T;H)} + \frac{1}{2}\,\mu_0\,\|\overline{v}\|^2_E + 2\|\overline{f}_{1,\varepsilon}\|^2_{L_1(0,T;H^*)} \\
&\qquad + \frac{2}{\mu_0}\|\overline{f}_2\|^2_{L_2(0,T;V^*)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{1}{2}\,\|\overline{v}\|^2_{L_\infty(0,T;H)} + 2k\|\overline{v}\|^2_{L_2(Q_T)} + \frac{1}{2}\,\mu_0\|\overline{v}\|_E \\
&\quad \le \|\overline{a}\|^2 + 2\|\overline{f}_{1,\varepsilon}\|^2_{L_1(0,T;H^*)} + \frac{2}{\mu_0}\|\overline{f}_2\|^2_{L_2(0,T;V^*)}.
\end{aligned}
$$

This estimate implies (7.4.1). □

Theorem 7.4.2. *For any solution $v \in W_1 \cap C_w([0,T];H)$ of problem (7.1.17), (7.1.18) with $\varepsilon \ge 0$, there is the estimate*

$$
\|v'\|_{L_1(0,T;V^*)} \le C\,(1 + \|f_{1,\varepsilon}\|_{L_1(0,T;H^*)} + \|f_2\|_{L_2(0,T;V^*)} + \|a\|)^2, \quad (7.4.5)
$$

with a constant C independent of ε and δ.

Proof. We repeat the arguments used in the proof of estimate (7.3.3). Express v' from equality (7.1.17): $v' = -(A(v) - K_\varepsilon(v) + C(v, Z_\delta(v)) - f_\varepsilon)$. Thus,

$$
\begin{aligned}
\|v'\|_{L_1(0,T;V^*)} \le C\,\big(&\|A(v)\|_{E^*} + \|K_\varepsilon(v)\|_{L_1(0,T;V^*)} \\
&+ \|C(v, Z_\delta(v))\|_{E^*} + \|f_{1,\varepsilon}\|_{L_1(0,T;H^*)} \\
&+ \|f_2\|_{L_2(0,T;V^*)}\big).
\end{aligned} \quad (7.4.6)
$$

Let us estimate $\|K_\varepsilon(v)\|_{L_1(0,T;V^*)}$. Since, for $n \le 4$, the embedding $V \subset L_4(\Omega)^n$ is continuous, for $u \in V$ we have

$$
\|K_\varepsilon(u)\|_{V^*} \le \max_{i,j}\Big\|\frac{u_i u_j}{1+\varepsilon|u|^2}\Big\|_{L_2(\Omega)} \le \max_{i,j}\|u_i u_j\|_{L_2(\Omega)} \le C\|u\|^2_{L_4(\Omega)^n}.
$$

Hence, $\|K_\varepsilon(v)\|_{L_1(0,T;V^*)} \le C\,\|v\|^2_{L_2(0,T;L_4(\Omega)^n)}$. Furthermore, the embedding $E = L_2(0,T;V) \subset L_2(0,T;L_4(\Omega)^n)$ is also continuous, so $\|v\|_{L_2(0,T;L_4(\Omega)^n)} \le C\|v\|_E$, and $\|K_\varepsilon(v)\|_{L_1(0,T;V^*)} \le C\|v\|^2_E$. This estimate and the properties of the maps A and C allow us to derive (7.4.5) from (7.4.1) and (7.4.6). □

Proof of Theorem 7.1.2. Take a sequence of positive numbers $\{\varepsilon_l\}$ converging to zero. For every number ε_l, the corresponding problem (7.1.17), (7.1.18) has at least one solution $v_l \in W$. Due to (7.1.16),

$$f_{1,\varepsilon_l} \to f_1 \;\text{ in }\; L_1(0,T;H^*) \;\text{ as }\; l \to \infty.$$

Hence, $\|f_{1,\varepsilon_l}\|_{L_1(0,T;H^*)}$ are uniformly bounded. Due to estimate (7.4.1), the sequence $\{\|v_l\|_{EL}\}$ is bounded. From estimate (7.4.5) it follows that the sequence of derivatives $\{v_l'\}$ is bounded in the space $L_1(0,T;V^*)$. Then without loss of generality we may assume that

$$\begin{aligned}
v_l &\rightharpoonup v^* && \text{weakly in } E;\\
v_l &\rightharpoonup v^* && \text{* - weakly in } L_\infty(0,T;H);\\
v_l &\to v^* && \text{strongly in } L_2(Q_T)^n;\\
v_l' &\rightharpoonup v^{*\prime} && \text{in the sense of distributions.}
\end{aligned}$$

Since linear operators are weakly continuous, $A(v_l) \rightharpoonup A(v^*)$ weakly in E^*, and $\mathcal{E}(v_l)(s,x) \rightharpoonup \mathcal{E}(v^*)(s,x)$ weakly in $L_2(Q_T,\mathbb{R}^{n\times n})$.

Let us show that

$$C(v_l, Z_\delta(v_l)) \rightharpoonup C(v^*, Z_\delta(v^*)) \quad \text{weakly in } E^*. \tag{7.4.7}$$

Due to Lemma 7.2.3,

$$Z_\delta(v_l) \to Z_\delta(v^*) \;\text{ in }\; C([0,T]\times[0,T], C(\overline{\Omega})^n). \tag{7.4.8}$$

Let $h \in E$ be an arbitrary function. Then

$$\begin{aligned}
&\big\langle C(v_l, Z_\delta(v_l)) - C(v^*, Z_\delta(v^*)), h\big\rangle\\
&\quad = \big\langle C(v_l, Z_\delta(v_l)) - C(v^*, Z_\delta(v_l)), h\big\rangle + \big\langle C(v^*, Z_\delta(v_l)) - C(v^*, Z_\delta(v^*)), h\big\rangle.
\end{aligned}$$

The second term converges to zero due to (7.4.8) and continuity of the map C in z. In the first term, let us change the variable x for $z = Z_\delta(v_l)(s,t,x)$. The inverse change of variable looks like $x = Z_\delta(v_l)(t,s,z)$.

$$\begin{aligned}
&\big\langle C(v_l, Z_\delta(v_l)) - C(v^*, Z_\delta(v_l)), h\big\rangle\\
&\quad = \int_0^T \int_\Omega \int_0^t \big[\mathcal{E}(v_l)(s, Z_\delta(v_l)(s,t,x)) - \mathcal{E}(v^*)(s, Z_\delta(v_l)(s,t,x))\big]\, ds\\
&\qquad\quad \cdot \mathcal{E}(h)(t,x)\, dx\, dt\\
&\quad = \int_0^T \int_\Omega \int_0^t \big[\mathcal{E}(v_l)(s,z) - \mathcal{E}(v^*)(s,z)\big]\, ds \cdot \big[\mathcal{E}(h)(t, Z_\delta(v_l)(t,s,z))\big]\, dz\, dt.
\end{aligned}$$

The first expression in square brackets converges to zero weakly in $L_2(Q_T, \mathbb{R}^{n\times n})$. Using continuity of the map Φ defined by equality (7.2.4), from the proof of Lemma 7.2.2 and (7.4.8) we conclude that the second expression converges strongly in $L_2([0,T]\times[0,T], L_2(\Omega, \mathbb{R}^{n\times n}))$. Then the triple integral converges to zero as $l \to \infty$, and this finishes the proof of (7.4.7).

Note that $K_{\varepsilon_l}(v_l) \to K(v^*)$ in the sense of distributions (cf. (6.4.8) with $\tau \equiv 0$). Passing to the limit in the sense of distributions as $l \to \infty$ in the equality

$$v_l' + A(v_l) - K_\varepsilon(v_l) + C(v_l, Z_\delta(v_l)) = f_{1,\varepsilon_l} + f_2,$$

we get equality (7.1.14) for the function v^*. Hence, v^* is a solution of equation (7.1.14). Note that as $v^* \in E$, equality (7.1.14) implies $v^{*'} \in E_1^*$ (cf. the proof of Theorem 7.4.2). Therefore, $v^* \in W_1$. Since $v^* \in L_\infty(0,T;H)$, Lemmas 2.2.3 and 2.2.6 yield $v^* \in C_w([0,T];H)$. Arguments as in the proof of Theorem 6.4.1 show that v^* satisfies the initial condition. □

7.5 Another weak formulation for the regularized Jeffreys model

Now we are going to study the problem of convergence of weak solutions of the initial-boundary value problems for the regularized model (7.1.10) – (7.1.13) to weak solutions of the initial-boundary value problem for system (6.2.1) – (6.2.5) as δ tends to zero.

Hereafter we assume that Ω is a sufficiently regular bounded domain in $\mathbb{R}^n$, $2 \leq n \leq 3$.

Let $a \in H$, $\sigma_0 \in W_2^{-1}(\Omega, \mathbb{R}_S^{n\times n})$, $f \in L_2(0,T;V^*)$. We recall (Definition 6.2.1) that a weak solution of problem (6.2.1) – (6.2.5) is a pair of functions (u,σ),

$$u \in L_2(0,T;V)\bigcap C_w([0,T];H),\ \frac{du}{dt} \in L_1(0,T;V^*), \tag{7.5.1}$$

$$\sigma \in L_2(0,T;L_2(\Omega,\mathbb{R}_S^{N\times N}))\bigcap C_w\big([0,T];H^{-1}\big(\Omega,\mathbb{R}_S^{N\times N}\big)\big) \tag{7.5.2}$$

satisfying the initial condition

$$u|_{t=0} = a,\ \sigma|_{t=0} = \sigma_0 \tag{7.5.3}$$

and the identities

$$\frac{d}{dt}(u,\varphi) + (\sigma, \nabla\varphi) - \sum_{i=1}^{n} \big(u_i u, \frac{\partial \varphi}{\partial x_i}\big) = \langle f, \varphi \rangle, \tag{7.5.4}$$

$$\begin{aligned}(\sigma, \Phi) + \lambda_1 \frac{d}{dt}(\sigma, \Phi) - \lambda_1 \sum_{i=1}^{n} \big(u_i \sigma,\ \frac{\partial \Phi}{\partial x_i}\big) \\ = 2\eta(\mathcal{E}(u), \Phi) + 2\eta\lambda_2 \Big(\frac{d}{dt}(\mathcal{E}(u), \Phi) - \sum_{i=1}^{n} \big(u_i \mathcal{E}(u), \frac{\partial \Phi}{\partial x_i}\big)\Big)\end{aligned} \tag{7.5.5}$$

for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$ in the sense of distributions on $(0, T)$. As in Chapter 6, C_0^∞ stands for the space $C_0^\infty(\Omega, \mathbb{R}_S^{n\times n})$.

Hereafter let us assume (7.1.7) again. Definition 7.1.1 implies that a weak solution of the regularized problem (7.1.10) – (7.1.13) is a function (let us denote it u here) from class (7.5.1) satisfying the initial condition

$$u|_{t=0} = a \tag{7.5.6}$$

and the identity

$$\begin{aligned}\frac{d}{dt}(u(t), \varphi) - \sum_{i=1}^{n} \big(u_i(t) u(t), \frac{\partial \varphi}{\partial x_i}\big) + \frac{1}{2}\mu_0 (\nabla u(t), \nabla \varphi) \\ + \mu \Big(\int_0^t e^{\frac{s-t}{\lambda_1}} \mathcal{E}(s, Z_\delta(u)(s,t,\cdot))\, ds, \mathcal{E}(\varphi) \Big) = \langle f(t), \varphi \rangle\end{aligned} \tag{7.5.7}$$

for all $\varphi \in V$ on $(0, T)$.

Let us also consider another problem which is obtained as an immediate regularization of problem (6.2.1) – (6.2.5). Let us define a weak solution of the problem as a pair of functions (u, σ) satisfying conditions (7.5.3), (7.5.4) and the equality

$$\begin{aligned}(\sigma, \Phi) + \lambda_1 \frac{d}{dt}(\sigma, \Phi) - \lambda_1 \sum_{i=1}^{n} \big((S_\delta u)_i \sigma,\ \frac{\partial \Phi}{\partial x_i}\big) \\ = 2\eta(\mathcal{E}(u), \Phi) + 2\eta\lambda_2 \Big(\frac{d}{dt}(\mathcal{E}(u), \Phi) - \sum_{i=1}^{n} \big((S_\delta u)_i \mathcal{E}(u), \frac{\partial \Phi}{\partial x_i}\big)\Big)\end{aligned} \tag{7.5.8}$$

for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$ in the sense of distributions on $(0, T)$.

It turns out that problems (7.5.6), (7.5.7) and (7.5.3), (7.5.4), (7.5.8) are associated as follows:

Theorem 7.5.1 (see [74]). *Let a function u from class (7.5.1) be a solution of problem (7.5.6), (7.5.7). Then the pair (u, σ) where*

$$\sigma(t) = \mu_0 \mathcal{E}(u)(t) + \mu \int_0^t e^{-\frac{t-s}{\lambda_1}} \mathcal{E}(u)(s, Z_\delta(u)(s,t,\cdot))\, ds, \tag{7.5.9}$$

is a solution of problem (7.5.3), (7.5.4), (7.5.8) in class (7.5.1), (7.5.2).

For the proof of Theorem 7.5.1 we will need two technical lemmas.
Denote by I_1 the second term in the right-hand side of (7.5.9).

Lemma 7.5.1. *The following estimate is valid:*

$$\|I_1\|_{L_\infty(0,T;L_2)} \le K(\|u\|_{L_2(0,T;V)}), \tag{7.5.10}$$

where the constant K does not depend on δ.

Proof. Note that the expression I_1 can be rewritten as $\mu B(u, Z_\delta(u))$. We recall that the operator B is defined by the formula

$$B(u,z)(t) = \int_0^t e^{-\frac{t-s}{\lambda_1}} \mathcal{E}(u)(s, z(s,t,\cdot))\,ds, \;\; u \in L_2(0,T;V), \quad z \in CG.$$

We have:

$$\begin{aligned}
\|B(u,z)\|^2_{L_\infty(0,T;L_2)} &= \sup_{t\in(0,T)} \int_\Omega \int_0^t \big[e^{-\frac{t-s}{\lambda_1}} |\mathcal{E}(u)(s,z(s,t,x))|\,ds\big]^2 dx \\
&\le \sup_{t\in(0,T)} \int_0^T \int_\Omega |\mathcal{E}(u)(s,z(s,t,x))|^2 dxds \\
&= \int_0^T \int_\Omega |\mathcal{E}(u)(s,z)|^2 dzds \\
&= \int_0^T \|\mathcal{E}(u)(s,\cdot)\|^2 ds \le K\|u\|^2_{L_2(0,T;V)}.
\end{aligned}$$

We have taken into account the identity $\det(\partial z/\partial x) = 1$. From these equalities and inequalities estimate (7.5.10) immediately follows. □

Lemma 7.5.2. *Let $v \in L_2(0,T;C^1(\overline{\Omega})^n \cap V)$, and z_v be a solution of equation (7.1.1). Then for every function $\zeta \in L_2(0,T;L_2(\Omega,\mathbb{R}))$ the integral*

$$\xi(t) = \int_0^t \zeta(s, z_v(s,t,\cdot))\,ds$$

belongs to $L_2(0,T;L_2)$ and the identity

$$\frac{d}{dt}(\xi,\psi) - \sum_{i=1}^n \big(v_i\xi, \frac{\partial\psi}{\partial x_i}\big) = (\zeta,\psi) \tag{7.5.11}$$

is valid for all $\psi \in C_0^\infty(\Omega,\mathbb{R})$ in the sense of distributions on $(0,T)$. Furthermore, $\frac{d\xi}{dt} \in L_1(0,T;H^{-2})$.

Proof. Take a sequence of functions $\{\zeta_m\}$ smooth enough and converging to ζ in $L_2(0,T;L_2)$. Then the corresponding ξ_m converge to ξ in $L_2(0,T;L_2)$. Really,

$$\begin{aligned}
&\|\xi_m - \xi\|^2_{L_2(0,T;L_2)} \\
&\quad = \int_0^T \Big\| \int_0^t \zeta_m(s, z_v(s,t,x)) - \zeta(s, z_v(s,t,x))\, ds \Big\|^2_{L_2(\Omega,\mathbb{R}^{n\times n})} dt \\
&\quad \leq \int_0^T \int_0^T \int_\Omega |(\zeta_m - \zeta)(s, z_v(s,t,x))|^2\, dx\, ds\, dt \\
&\quad = T \int_0^T \int_\Omega |(\zeta_m - \zeta)(s,z)|^2\, dz\, ds = T\|\zeta_m - \zeta\|^2_{L_2(0,T;L_2)} \\
&\quad \to 0.
\end{aligned}$$

But for the smooth functions ξ_m and ζ_m formula (1.3.9) gives

$$\frac{d}{dt}\xi_m = \zeta_m.$$

Here (only here!) $\frac{d}{dt}$ stands for the substantial derivative, i.e.

$$\frac{\partial \xi_m(t)(x)}{\partial t} + \sum_{i=1}^n v_i \frac{\partial \xi_m(t)(x)}{\partial x_i} = \zeta_m(t)(x).$$

Taking the $L_2(\Omega,\mathbb{R})$-scalar product of this equality with an arbitrary function $\psi \in C_0^\infty(\Omega,\mathbb{R})$ at almost all $t \in (0,T)$, and integrating by parts in the second term of the left-hand side, we arrive at

$$\frac{d}{dt}(\xi_m, \psi) - \sum_{i=1}^n \big(v_i \xi_m, \frac{\partial \psi}{\partial x_i}\big) = (\zeta_m, \psi).$$

Passing to the limit as $m \to \infty$ in the sense of distributions on $(0,T)$ we obtain formula (7.5.11).

We have from (7.5.11), using Hölder's inequality:

$$\begin{aligned}
\|\langle \xi', \psi\rangle\|_{L_1(0,T)} &= \Big\|\frac{d}{dt}(\xi,\psi)\Big\|_{L_1(0,T)} \leq \sum_{i=1}^n \Big\|\big(v_i \xi, \frac{\partial\psi}{\partial x_i}\big)\Big\|_{L_1(0,T)} + \|(\zeta,\psi)\|_{L_1(0,T)} \\
&\leq \sum_{i=1}^n \|v_i\|_{L_2(0,T;L_4)} \|\xi\|_{L_2(0,T;L_2)} \Big\|\frac{\partial\psi}{\partial x_i}\Big\|_{L_4} + \|\zeta\|_{L_2(0,T;L_2)}\|\psi\|_{L_2} \\
&\leq \|v\|_{L_2(0,T;V)}\|\xi\|_{L_2(0,T;L_2)}\|\psi\|_{H_0^2} + \|\zeta\|_{L_2(0,T;L_2)}\|\psi\|_{L_2} \leq C\|\psi\|_{H_0^2}.
\end{aligned}$$

Here we have used the continuity of embedding of $H_0^1(\Omega)$ and V into L_4. Thus, $\frac{d\xi}{dt} \in L_1(0,T;H^{-2})$. □

Proof of Theorem 7.5.1. Let us show first that the defined by formula (7.5.9) function σ belongs to class (7.5.2). Really, since $u \in L_2(0,T;V) \bigcap C_w([0,T];H)$, the first term in (7.5.9) $\mu_0 \mathcal{E}(u) \in L_2(0,T;L_2) \bigcap C_w([0,T];H^{-1})$. By Lemma 7.5.1 the second term in (7.5.9) $I_1 \in L_\infty(0,T;H)$. And from Lemma 7.5.2 it follows that $\frac{d}{dt} I_1 \in L_1(0,T;H^{-2})$. Then, by Lemma 2.2.3, $I_1 \in C([0,T];H^{-2})$, so, by Lemma 2.2.6, $I_1 \in C_w([0,T];H)$. Hence, σ belongs to class (7.5.2).

From (7.5.7) and (7.5.9) it immediately follows that (7.5.4) is fulfilled. And (7.5.9) and (7.1.7) yield

$$\sigma|_{t=0} = \mu_0 \mathcal{E}(u)|_{t=0} = \mu_0 \mathcal{E}(a) = \sigma_0.$$

Thus, the initial condition (7.5.3) is fulfilled.

It remains to show (7.5.8). Take an arbitrary $\Phi \in C_0^\infty$. We have from (7.5.9):

$$e^{\frac{t}{\lambda_1}}(\sigma - \mu_0 \mathcal{E}(u)) = \mu \int_0^t e^{\frac{s}{\lambda_1}} \mathcal{E}(u)(s, Z_\delta(u)(s,t,\cdot))\, ds.$$

By Lemma 7.5.2 we have

$$\frac{d}{dt}(e^{\frac{t}{\lambda_1}}(\sigma - \mu_0 \mathcal{E}(u)), \Phi) - e^{\frac{t}{\lambda_1}} \sum_{i=1}^n \Big((S_\delta(u))_i (\sigma - \mu_0 \mathcal{E}(u)), \frac{\partial \Phi}{\partial x_i}\Big) = \mu e^{\frac{t}{\lambda_1}} (\mathcal{E}(u), \Phi),$$

whence

$$\begin{aligned} &\frac{1}{\lambda_1} e^{\frac{t}{\lambda_1}} (\sigma - \mu_0 \mathcal{E}(u), \Phi) + e^{\frac{t}{\lambda_1}} \frac{d}{dt} (\sigma - \mu_0 \mathcal{E}(u), \Phi) \\ &\quad - e^{\frac{t}{\lambda_1}} \sum_{i=1}^n \Big((S_\delta(u))_i (\sigma - \mu_0 \mathcal{E}(u)), \frac{\partial \Phi}{\partial x_i}\Big) = \mu e^{\frac{t}{\lambda_1}} (\mathcal{E}(u), \Phi). \end{aligned}$$

Dividing by $\frac{1}{\lambda_1} e^{\frac{t}{\lambda_1}}$ and rearranging the terms, we obtain:

$$\begin{aligned} &(\sigma, \Phi) + \lambda_1 \frac{d}{dt}(\sigma, \Phi) - \lambda_1 \sum_{i=1}^n \Big((S_\delta u)_i \sigma, \frac{\partial \Phi}{\partial x_i}\Big) \\ &= (\mu_0 + \mu \lambda_1)(\mathcal{E}(u), \Phi) + \mu_0 \lambda_1 \frac{d}{dt} (\mathcal{E}(u), \Phi) - \mu_0 \lambda_1 \sum_{i=1}^n \Big((S_\delta u)_i \mathcal{E}(u), \frac{\partial \Phi}{\partial x_i}\Big). \end{aligned}$$

This immediately yields (7.5.8). $\square$

7.6 Behaviour of solutions of regularized problems as $\delta \to 0$

In this section we assume $a \in H$, $\sigma_0 \in W_2^{-1}(\Omega, \mathbb{R}_S^{n\times n})$, $f \in L_2(0,T;V^*)$ and condition (7.1.7) to be fulfilled. Theorems 7.1.2, 7.4.1, 7.4.2 guarantee existence of solutions for problem (7.5.6), (7.5.7) which satisfy the estimate

$$\|u\|_{L_2(0,T;V)} + \|u\|_{L_\infty(0,T;H)} + \|\frac{du}{dt}\|_{L_1(0,T;V^*)} \leq K(\|a\|, \|f\|_{L_2(0,T;V^*)}), \tag{7.6.1}$$

where the constant K does not depend on δ. By Theorem 7.5.1 these solutions generate pairs of solutions for problem (7.5.3), (7.5.4), (7.5.8). Now we are interested in behavior of these solutions as $\delta \to 0$. The following theorem gives an answer to this question.

Theorem 7.6.1 (see [74]). *Let $\delta_k \to 0$ for $k \to \infty$ be a sequence of numbers, u_k be some solutions of problem (7.5.6), (7.5.7) at corresponding δ_k, satisfying estimate (7.6.1), and let σ_k be the corresponding deviatoric stress tensors given by formula (7.5.9). Then it is possible to select a subsequence $\delta_{k_m} \to 0$ for $m \to \infty$ and a pair (u_*, σ_*) so that*

the pair (u_*, σ_*) is a weak solution for problem (6.2.1) – (6.2.5);

$$u_{k_m} \to u_* \text{*-weakly in } L_\infty(0,T;H), \text{weakly in } L_2(0,T;V),$$
$$\text{strongly in } L_2(0,T;H);$$
$$\sigma_{k_m} \to \sigma_* \text{*-weakly in } L_\infty(0,T;H^{-1}), \text{weakly in } L_2(0,T;L_2).$$

Proof. From estimate (7.6.1) it follows that there exist a subsequence δ_{k_m} of sequence δ_k and a function u_* so that

$$u_{k_m} \to u_* \text{ weakly in } L_2(0,T;V),$$
$$u_{k_m} \to u_* \text{ *-weakly in } L_\infty(0,T;H)$$

and the sequence $\|\frac{du_{k_m}}{dt}\|_{L_1(0,T;V^*)}$ is bounded. Due to Theorem 2.2.6, without loss of generality we may assume that

$$u_{k_m} \to u_* \text{ strongly in } L_2(0,T;H).$$

So we have:

$$\mathcal{E}(u_{k_m}) \to \mathcal{E}(u_*) \text{ weakly in } L_2(0,T;L_2),$$
$$\mathcal{E}(u_{k_m}) \to \mathcal{E}(u_*) \text{ *-weakly in } L_\infty(0,T;H^{-1}).$$

By Lemma 7.5.1 the expressions

$$\mu \int_0^t e^{-\frac{t-s}{\lambda_1}} \mathcal{E}(u_{k_m})(s, Z_{\delta_{k_m}}(u_{k_m})(s,t,x))\, ds$$

are bounded in $L_\infty(0,T;L_2)$ and, therefore, in $L_\infty(0,T;H^{-1})$ and $L_2(0,T;L_2)$.

Then from (7.5.9) it follows that without loss of generality there is σ_* such that

$$\sigma_{k_m} \to \sigma_* \text{ weakly in } L_2(0,T;L_2),$$
$$\sigma_{k_m} \to \sigma_* \text{ *-weakly in } L_\infty(0,T;H^{-1}).$$

The operators S_δ converge strongly to the identity operator in $L_2(0,T;H)$ as $\delta \to 0$; therefore $\|S_\delta\| \le K$, where K does not depend on δ.

We have:

$$\|S_{\delta_{k_m}} u_{k_m} - u_*\|_{L_2(0,T;H)} \le \|S_{\delta_{k_m}}\| \, \|u_{k_m} - u_*\|_{L_2(0,T;H)} + \|(S_{\delta_{k_m}} - I)u_*\|_{L_2(0,T;H)}.$$

Both terms vanish as $m \to \infty$. Thus,

$$S_{\delta_{k_m}} u_{k_m} \to u_* \text{ strongly in } L_2(0,T;H).$$

Fix an arbitrary function $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$. We have:

$$\begin{aligned}
&\int_0^T \Big| \sum_{i=1}^n \Big[\big((S_{\delta_{k_m}} u_{k_m})_i \sigma_{k_m}, \frac{\partial \Phi}{\partial x_i}\big) - \big(u_{*i}\sigma_*, \frac{\partial \Phi}{\partial x_i}\big) \Big] \Big| \, dt \\
&\le \int_0^T \sum_{i=1}^n \Big[\Big| \big([(S_{\delta_{k_m}} u_{k_m})_i - u_{*i}]\sigma_{k_m}, \frac{\partial \Phi}{\partial x_i}\big) \Big| + \Big| \big(u_{*i}[\sigma_{k_m} - \sigma_*], \frac{\partial \Phi}{\partial x_i}\big) \Big| \Big] \, dt \\
&\le \sum_{i=1}^n \Big[\|S_{\delta_{k_m}} u_{k_m} - u_*\|_{L_2(0,T;H)} \|\sigma_{k_m}\|_{L_2(0,T;L_2)} \Big\| \frac{\partial \Phi}{\partial x_i} \Big\|_{L_\infty} \\
&\qquad + \big\langle \sigma_{k_m} - \sigma_*, u_{*i} \frac{\partial \Phi}{\partial x_i} \big\rangle_{L_\infty(0,T;L_2) \times L_1(0,T;L_2)} \Big].
\end{aligned}$$

All terms converge to zero. Therefore

$$\sum_{i=1}^n \big((S_{\delta_{k_m}} u_{k_m})_i \sigma_{k_m}, \frac{\partial \Phi}{\partial x_i}\big) \to \sum_{i=1}^n \big(u_{*i}\sigma_*, \frac{\partial \Phi}{\partial x_i}\big) \text{ in } L_1(0,T).$$

Similarly

$$\sum_{i=1}^n \big((S_{\delta_{k_m}} u_{k_m})_i \mathcal{E}(u_{k_m}), \frac{\partial \Phi}{\partial x_i}\big) \to \sum_{i=1}^n \big(u_{*i}\mathcal{E}(u_*), \frac{\partial \Phi}{\partial x_i}\big) \text{ in } L_1(0,T).$$

It is easy to see that the remaining terms in (7.5.4) and (7.5.8) with $u_{k_m}, \sigma_{k_m}, \delta_{k_m}$ substituted into these equalities converge to corresponding terms of (7.5.4) and (7.5.5) with u_*, σ_* in the sense of distributions on $(0, T)$. Therefore the pair (u_*, σ_*) satisfies (7.5.4), (7.5.5). As in the proof of Theorem 6.4.1, one can check that the initial condition (7.5.3) is fulfilled. Furthermore, as in the proof of estimates (6.3.14) and (6.5.2) one can establish that

$$\frac{du_*}{dt} \in L_1(0, T; V^*), \frac{d}{dt}(\sigma_* - \mu_0 \mathcal{E}(u_*)) \in L_2(0, T; H^{-2}).$$

By Lemma 2.2.3, $u_* \in C([0, T]; V^*), \sigma_* - \mu_0 \mathcal{E}(u_*) \in C([0, T]; H^{-2})$. Since $u_* \in L_\infty(0, T; H)$ and $\sigma_* - \mu_0 \mathcal{E}(u_*) \in L_\infty(0, T; H^{-1})$, Lemma 2.2.6 implies that $u_* \in C_w([0, T]; H)$, $\sigma_* - \mu_0 \mathcal{E}(u_*) \in C_w([0, T]; H^{-1})$. This yields $\mathcal{E}(u_*) \in C_w([0, T]; H^{-1})$ and $\sigma_* \in C_w([0, T]; H^{-1})$. $\square$

7.7 Two constructions of regularization operator

7.7.1 The first construction

Let Ω be a sufficiently regular bounded domain in $\mathbb{R}^n$.

For $1 \le q < \infty$, denote the closure of the set $\mathcal{V}(\Omega)$ in the norm of the space $L_q(\Omega)^n$ by H_q, and the closure of $\mathcal{V}(\Omega)$ in the norm of the space $W_q^1(\Omega)^n$ by V_q.

In this section we construct the regularization operator S_δ (see [17, 18]). We recall (Section 7.1) that it should satisfy the properties: i) the operator $S_\delta : H \to C^1(\overline{\Omega})^n \cap V$ for $\delta > 0$; ii) the map $S_\delta : L_2(0, T; H) \to L_2(0, T; C^1(\overline{\Omega})^n \cap V)$ generated by this operator is continuous; iii) the operators $S_\delta : L_2(0, T; H) \to L_2(0, T; H)$ converge strongly to the identity operator I as $\delta \to 0$. However, we are going to present more general constructions which are suitable not only for L_2-case but also for L_q-case. Namely, we are constructing an operator $S_\delta : H_q \to C^1(\overline{\Omega})^n \cap V_q$ such that the map $S_\delta : L_q(0, T; H_q) \to L_q(0, T; C^1(\overline{\Omega})^n \cap V_q)$ is continuous, and $S_\delta(v)$ converges to v in $L_q(0, T; H_q)$ as $\delta \to 0$ for all $v \in L_q(0, T; V_q)$ (and even for $v \in L_2(0, T; H)$ if $q = 2$).

We point out that classical averaging procedures are not directly applicable here since they do not conserve the condition $v(t) \in V_q$.

Choose a finite covering of the domain Ω by sets $U, U_1, \dots, U_k$ such that i) $U \subset \overline{U} \subset \Omega$, ii) each of the sets $\partial\Omega \cap U_j$, $j = 1, 2, \dots, k$, is not empty and is a graph of a Lipschitz function, iii) each of the sets $\Omega \cap U_j$ is star-shaped with respect to one of its points x_j^*.

Choose a C^∞-smooth partition of unity subordinated to the covering $U, U_1, \dots, U_k$ of the set Ω, i.e. scalar functions ϕ, ϕ_j, $j = 1, 2, \dots, k$, on Ω such that

$$\phi + \sum_{j=1}^{k} \phi_j = 1, \quad \text{where} \quad \operatorname{supp}\phi \subset U, \quad \operatorname{supp}\phi_j \subset U_j, \quad j = 1, 2, \dots, k.$$

Then for a function $v \in H_q$ we have

$$v = \phi v + \sum_{j=1}^{k} \phi_j v, \tag{7.7.1}$$

and each term belongs to $L_q(\Omega)^n$.

Let $\sigma_\varepsilon(x_j^*)$, $\varepsilon \neq 0$, denote the homothetic transformation with coefficient $1-\varepsilon$:

$$x \mapsto x_j^* \cdot \varepsilon + x \cdot (1-\varepsilon).$$

For this transformation $\sigma_\varepsilon(\Omega \cap U_j) \subset \sigma_\varepsilon(\overline{\Omega} \cap U_j) \subset \Omega \cap U_j$ for every j. Denote by u_j the function $u_j = \phi_j v$ and by $\sigma_\varepsilon[u_j]$ the function $\sigma_\varepsilon[u_j](x) = u_j(\sigma_\varepsilon(x))$. We assume that each of the functions u_j is continued by zero onto the whole space $\mathbb{R}^n$. Due to [61, Chapter I, Lemma 1.1], $\sigma_\varepsilon[u_j]$ converges to u_j in $L_q(\Omega)^n$ as $\varepsilon \to 0$. Note also that for fixed $\varepsilon > 0$ the support of each function $\sigma_\varepsilon[u_j]$, $j = 1, 2, \ldots, k$, and its 2δ-neighbourhood is contained in $U_j \cap \Omega$ for $\delta > 0$ small enough. The same is true for the function $u = \phi v$, since the function ϕ has a compact support in Ω. Thus, the function

$$\overline{v} = u + \sum_{j=1}^{k} \sigma_\varepsilon[u_j]$$

has a compact support in Ω.

Take the subdomain of the domain Ω of the form

$$\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\},$$

containing the support of the function $\overline{v}$.

It is known (see [6, 36]) that the space $L_q(\Omega_\delta)^n$ may be decomposed in the following direct sum: $L_q(\Omega_\delta)^n = H_q(\Omega_\delta) \oplus G_q(\Omega_\delta)$, where $G_q(\Omega_\delta) = \{\nabla p : p \in W_q^1(\Omega_\delta)\}$. Denote by P_δ the operator of projection from $L_q(\Omega_\delta)^n$ onto $H_q(\Omega_\delta)$. Applying this operator to the function $\overline{v}$, one obtains $v_\delta = P_\delta(\overline{v})$. The function v_δ belongs to space $H_q(\Omega_\delta)$, therefore its continuation by zero onto the domain Ω has the property $\operatorname{div} v_\delta = 0$ in every point of the domain. Hence, $v_\delta \in H_q = H_q(\Omega)$.

Let ρ be a function of the class C^∞ with a compact support in the ball $B_1(0)$ of radius 1 centered at the origin, such that $\rho \geq 0$, $\int_{\mathbb{R}^n} \rho(x)dx = \int_{B_1(0)} \rho(x)dx = 1$. Denote by ρ_δ the function $\frac{1}{\delta^n}\rho(\frac{x}{\delta})$. As $\delta \to 0$ the functions ρ_δ converge in the sense of distributions to Dirac function and $\rho_\delta * v \to v$ in $L_q(\mathbb{R}^n)^n$ for any function $v \in L_q(\mathbb{R}^n)^n$, where $*$ is the convolution of functions.

Applying the operation of Steklov averaging to this function, we obtain $\widetilde{v} = \rho_\delta * v_\delta$. The choice of δ ensures the condition that the support of $\widetilde{v}$ is a compact set in Ω. Since

$$\operatorname{div} \widetilde{v} = \operatorname{div}(\rho_\delta * v_\delta) = \rho_\delta * \operatorname{div} v_\delta = 0,$$

one has $\widetilde{v} \in V \cap C^{\infty}(\overline{\Omega})^n$.

Each of the transformations used at the construction of $\widetilde{v}$ determines a linear bounded map in the corresponding spaces. Therefore the map $S_\delta : v \to \widetilde{v} = \rho_\delta * v_\delta$ satisfies the conditions on a regularization operator. Namely, $S_\delta : H_q \to V_q \cap C^1(\overline{\Omega})^n$ is continuous. Since the construction of S_δ does not depend on t, the map $S_\delta : L_q(0, T; H_q) \to L_q(0, T; V_q \cap C^1(\overline{\Omega})^n)$ is also continuous.

Besides, $S_\delta v \to v$ in $L_q(0, T; H_q)$ as $\delta \to 0$. The check of this fact is rather long, so we shall consider another construction of a regularization operator and present a proof of the convergence only for the second construction.

7.7.2 The second construction

Let Ω be a sufficiently regular bounded domain in $\mathbb{R}^n$. For $\delta > 0$, we denote by Ω_δ the set

$$\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

Let $v \in H_q$ be an arbitrary function. Denote by v_δ the restriction of the function v to the domain Ω_δ: $v_\delta = v\,|_{\Omega_\delta}$. We suppose that the function v_δ is continued by zero onto $\Omega \backslash \Omega_\delta$.

We recall that by P_δ we denote the operator of projection from $L_q(\Omega_\delta)^n$ onto $H_q(\Omega_\delta)$. Applying this operator to the function v_δ, we obtain $\overline{v} = P_\delta(v_\delta)$. The function $\overline{v}$ belongs to the space $H_q(\Omega_\delta)$, therefore its continuation by zero onto the domain Ω has the property $\overline{v} \in H_q = H_q(\Omega)$. Hence, $\operatorname{div} \overline{v} = 0$ in every point of the domain. Really, by definition of the space $H_q(\Omega_\delta)$ the function $\overline{v}$ is a limit of a sequence of functions $w_k \in \mathcal{V}(\Omega_\delta)$ in the norm of the space $L_q(\Omega_\delta)$. Continuing the functions w_k by zero on the complement of the domain Ω_δ, we obtain functions $\overline{w}_k \in \mathcal{V}(\Omega)$. It is easy to see that the function $\overline{v}$ is a limit of the sequence of functions $\overline{w}_k$ in the norm of the space $L_q(\Omega)$. Therefore this function belongs to the space $H_q(\Omega)$.

Again, let ρ be a function of the class C^∞ with a compact support in the ball of radius 1 centered at the origin, such that $\rho \geq 0$, $\int_{\mathbb{R}^n} \rho(x)dx = \int_{B_1(0)} \rho(x)dx = 1$. Denote by ρ_δ the function $\frac{2^n}{\delta^n}\rho(\frac{2x}{\delta})$.

Applying the operation of Steklov averaging to the function $\overline{v}$ we obtain $\widetilde{v} = \rho_\delta * \overline{v}$.

The choice of δ ensures the condition that the support of $\widetilde{v}$ is a compact set in Ω. Since

$$\operatorname{div} \widetilde{v} = \operatorname{div}(\rho_\delta * \overline{v}) = \rho_\delta * \operatorname{div} \overline{v} = 0,$$

one has $\widetilde{v} \in V \cap C^{\infty}(\overline{\Omega})^n$.

Each of the transformations used at the construction of $\widetilde{v}$ determines a linear bounded map in the corresponding spaces. Therefore the map $S_\delta : v \to \widetilde{v} = \rho_\delta * v_\delta$ satisfies the conditions on a regularization operator. Namely, $S_\delta : H_q \to V_q \cap C^1(\overline{\Omega})^n$ is continuous. Since the construction of S_δ does not depend on t, the map $S_\delta : L_q(0, T; H_q) \to L_q(0, T; V_q \cap C^1(\overline{\Omega})^n)$ is also continuous.

It is easy to see that the maps $S_\delta : V_q \to V_q \cap C^1(\overline{\Omega})^n$ and $S_\delta : L_q(0, T; V_q) \to L_q(0, T; V_q \cap C^1(\overline{\Omega})^n)$ are correctly defined and continuous.

Let us check that for any function $v \in V_q$ (or $v \in H$ for $q = 2$) one has

$$S_\delta v \to v \quad \text{in} \quad H_q \quad \text{as} \quad \delta \to 0.$$

First of all, it is easy to see that $\|v\ |_{\Omega\backslash\Omega_\delta}\ \|_{L_q(\Omega\backslash\Omega_\delta)^n} \to 0$ as $\delta \to 0$. It follows from the property of absolute continuity of the Lebesgue integral.

Let p be a solution of the boundary value problem

$$\begin{cases} \Delta p = 0, \\ \frac{\partial p}{\partial \nu_\delta}|_{\partial\Omega_\delta} \ = \ \gamma_{\nu_\delta}(v_\delta), \end{cases}$$

such that $\int_\Omega p(x)dx = 0$, where ν_δ is the exterior normal vector to the boundary $\partial\Omega_\delta$ of the domain Ω_δ and $\gamma_{\nu_\delta}(v_\delta) = v_\delta\ |_{\partial\Omega_\delta} \cdot \nu_\delta$ (this has a special meaning for $q = 2$ and $v \in H$, see [61]). Then $\overline{v} = P_\delta v_\delta = v_\delta - \operatorname{grad} p$ (cf. [61]).

Since

$$\|p\|_{W_q^1(\Omega_\delta)^n} \le C_0 \|\gamma_{\nu_\delta}(v_\delta)\|_{B_q^{1-1/q}(\partial\Omega_\delta)^n}$$

with some constant C_0 independent from δ, one has $\|p\|_{W_q^1(\Omega_\delta)^n} \to 0$ as $\delta \to 0$, if $\|\gamma_{\nu_\delta}(v_\delta)\|_{B_q^{1-1/q}(\partial\Omega_\delta)^n} \to 0$ as $\delta \to 0$ ($B_q^{1-1/q}$ is the Besov space [11]; it must be replaced by $H^{-1/2}$ if $q = 2$ and $v \in H$).

The concept of trace of a function assumes "continuous dependence" of values of a function on the variation of the manifold $\partial\Omega$. For small δ the manifolds $\partial\Omega_\delta$ and $\partial\Omega$ are close and there exists a one-to-one map of one manifold onto another, therefore the traces of a function on these manifolds $\gamma_{\nu_\delta}(v)$ and $\gamma_\nu(v)$ are also close. Since $\gamma_\nu(v) = 0$, one has $\|\gamma_{\nu_\delta}(v_\delta)\|_{B_q^{1-1/q}(\partial\Omega_\delta)^n}$ is close to zero, i.e. $\|\gamma_{\nu_\delta}(v_\delta)\|_{B_q^{1-1/q}(\partial\Omega_\delta)^n} \to 0$ as $\delta \to 0$. Thus, p tends to zero in W_q^1 as $\delta \to 0$.

Finally, the operation of Steklov averaging has the property

$$\rho_\delta * v \ \to \ v \ \text{ in } \ L_q(\mathbb{R}^n)^n$$

for any function $v \in L_q(\mathbb{R}^n)^n$. Therefore

$$\|\widetilde{v} - v\|_{L_q(\Omega)^n} \ \le \ \|\rho_\delta * \overline{v} - \rho_\delta * v\|_{L_q(\Omega)^n} + \|\rho_\delta * v - v\|_{L_q(\Omega)^n},$$

and the second term tends to zero. For the first term we have the estimate

$$\begin{aligned} \|\rho_\delta * \overline{v} - \rho_\delta * v\|_{L_q(\Omega)^n} \ &\le \ C_1 \|\overline{v} - v\|_{L_q(\Omega)^n} \\ &\le C_1(\|\operatorname{grad} p\|_{L_q} + \|v\ |_{\Omega\backslash\Omega_\delta}\ \|_{L_q(\Omega\backslash\Omega_\delta)^n}) \\ &\le C_1(\|p\|_{W_q^1} + \|v\ |_{\Omega\backslash\Omega_\delta}\ \|_{L_q(\Omega\backslash\Omega_\delta)^n}) \end{aligned}$$

with C_1 independent from δ. Both terms in the parentheses vanish as $\delta \to 0$. Thus, $\|\widetilde{v} - v\|_{L_q(\Omega)^n} \to 0$ as $\delta \to 0$.

Since $S_\delta v = \widetilde{v}$, for every function $v \in V_q$ (or H for $q = 2$) we have

$$S_\delta v \to v \quad \text{in} \quad H_q \quad \text{as} \quad \delta \to 0.$$

Moreover, if $v \in L_q(0, T; V_q)$ (or $v \in L_2(0, T; H)$ for $q = 2$), a similar reasoning shows that

$$S_\delta v \to v \quad \text{in} \quad L_q(0, T; H_q) \quad \text{as} \quad \delta \to 0.$$

References

[1] R. A. Adams, Sobolev spaces, Acad. Press, New York–San Francisco–London, 1975.

[2] Ju. Ja. Agranovich, P. E. Sobolevskii, Motion of a nonlinear viscoelastic fluid, DAN SSSR, 1990, V. 314, no. 3, 521–525.

[3] Yu. Ya. Agranovich, P. E. Sobolevskii, Motion of nonlinear viscoelastic fluid, Nonlinear Anal. TMA, #6 (1998), V. 32, 755–760.

[4] Y. Y. Agranovich, V. G. Zvyagin, Investigations of Properties of Attractors for a Regularized Model of the Motion of a Nonlinear-Viscous Fluid, International Journal of Mathematics, Game Theory and Algebra, 12(1) (2001), 71–87.

[5] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, B. N. Sadovskii, Measures of Noncompactness and Condensing Operators, Birkhäuser, Boston, Basel, Berlin, 1992.

[6] H. Amann, On the strong solvability of the Navier–Stokes equations, J. Math. Fluid Mech., 2 (2000), 16–98.

[7] G. Astarita, G. Marucci, Principles of non-Newtonian fluid hydromechanics, McGraw Hill, New York, 1974.

[8] J. P. Aubin, Un théorème de compacité, C.R. Acad. Sci., V. 256 (1963), 5042–5044.

[9] A. V. Babin, M. I. Vishik, Attractors of evolutionary equations, Nauka, Moscow, 1989.

[10] G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, 2000.

[11] O. V. Besov, V. P. Il'in, S. M. Nikol'skii, Integral representations of functions and imbedding theorems, Scripta Series in Mathematics, Edited by Mitchell H. Taibleson, V. H. Winston and Sons, Washington, D.C., Halsted Press [John Wiley and Sons], New York–Toronto, Ont.–London, 1978–1979.

[12] D. Cheban, Global Attractors of Nonautonomous Dissipative Dynamical Systems, Interdisciplinary Mathematical Sciences 1. River Edge, NJ, World Scientific, 2004.

[13] J.-Y. Chemin, N. Masmoudi, About lifespan of regular solutions of equations related to viscoelastic fluids, SIAM J. Math. Anal. 33 (2001), no. 1, 84–112.

[14] V. V. Chepyzhov, M. I. Vishik, Attractors of non-autonomous dynamical systems and their dimension, J. Math Pures Appl. 73 (1994), 279–333.

[15] V. V. Chepyzhov, M. I. Vishik, Evolution equations and their trajectory attractors, J. Math Pures Appl. 76 (1997), 913–964.

[16] V. T. Dmitrienko, V. G. Zvyagin, The topological degree method for equations of the Navier–Stokes type, Abstract and Applied Analysis, V. 2 (1997), No. 1–2, 1–45.

[17] V. T. Dmitrienko, V. G. Zvyagin, Investigation of a regularized model of motion of a viscoelastic medium, in: Analytical Approaches to Multidimensional Balance Laws, O. Rozanova (Ed.), Nova Science, New York, 2004.

[18] V. T. Dmitrienko, V. G. Zvyagin, Construction of a regularization operator in models of viscoelastic media motion, Vestnik VSU, Ser. phys.-math., no.2 (2004), 148–153 (in Russian).

[19] Yu A. Dubinskii, Weak convergence for nonlinear elliptic and parabolic equations, Mat. Sb., 67 (109) (1965), No. 4, 609–642.

[20] H. Gajewski, K. Groeger, K. Zacharias, Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen, Akademie-Verlag, Berlin, 1974.

[21] C. Guillopé, J. C. Saut, Existence results for the flow of viscoelastic fluids with a differential constitutive law, Nonlinear Anal. TMA, # 9 (1990), V.15, 849–869.

[22] C. Guillopé, J.-C. Saut, Mathematical problems arising in differential models for viscoelastic fluids, Mathematical topics in fluid mechanics (Lisbon, 1991), 64–92, Pitman Res. Notes Math. Ser., 274, Longman Sci. Tech., Harlow, 1992.

[23] I. Gyarmati Non-equilibrium thermodynamics. Field theory and variational principles, Springer, Heidelberg, 1970.

[24] E. Fernández Cara, F. Guillén, R. R. Ortega, Existence et unicité de solution forte locale en temps pour des fluides non Newtoniens de type Oldroyd (version L_r - L_s), C.R. Academie des Sciences Paris, tome 319, serie I, pp. 411–416.

[25] E. Fernández Cara, F. Guillén, R. R. Ortega, Some theoretical results concerning non-Newtonian fluids of the Oldroyd kind, Ann. Scuola Norm. Sup. Pisa Cl. Sci., V. 26 (1998), 1–29.

[26] A. Hakim, Mathematical analysis of viscoelastic fluids of White-Metzner type, J. Math. Anal. Appl. #3 (1994), V. 185, 675–705.

[27] J. Hale, Asymptotic behavior of dissipative systems, Mathematical surveys and Monographs, 25, American Math. Soc., Providence, R.I., 1988.

[28] A. Haraux, Systèmes dynamiques dissipatifs et applications, Masson, Paris, 1991.

[29] R. H. W. Hoppe, M. Y. Kuzmin, W. G. Litvinov, and V. G. Zvyagin, Flow of electrorheological fluid under conditions of slip on the boundary, Abstract and Applied Analysis, vol. 2006 (2006), Article ID 43560, 14 pages.

[30] A. A. Kotsiolis, A. P. Oskolkov, On the limiting conditions and an attractor for the motion equations of Oldroyd's fluid, Zap. nauch. sem. LOMI, V. 152 (1986), 67–71.

[31] M. Krasnoselskii, Topological methods in the theory of nonlinear integral equations, Gostehizdat, 1956, (Russian); Engl. transl., Macmillan, 1964.

[32] M. A. Krasnosel'skii, P. P. Zabreiko, Geometrical methods of nonlinear analysis, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 263. Springer-Verlag, Berlin, 1984.

[33] M. A. Krasnosel'skii, P. P. Zabreiko, E. I. Pustyl'nik, P. E. Sobolevskii, Integral operators in spaces of summable functions, Monographs and Textbooks on Mechanics of Solids and Fluids, Mechanics: Analysis, Noordhoff International Publishing, Leiden, 1976.

[34] S. G. Krein and Y. I. Petunin, Scales of Banach spaces, Russ. Math. Surv., V. 21 (1966), no. 2, 85–159.

[35] O. A. Ladyzenskaya, On the dynamic system generated by Navier–Stokes equations, Zap. nauch. sem. LOMI, V. 27 (1972), 91–115.

[36] O. A. Ladyzenskaya, The Mathematical Theory of Viscous Incompressible Flow, New York: Gordon and Breach, 2nd edition (1963).

[37] J.-L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969.

[38] V. G. Litvinov, Motion of the nonlinear-viscous fluid, Moscow, Nauka, 1982, (Russian).

[39] W. G. Litvinov, A model and general problem on plastic flow under deformations, Universität Stuttgart, Bericht, 1999, 99/07.

[40] N. G. Lloyd, Degree theory, Cambridge University Press, 1978.

[41] E. Magenes and G. Stampacchia, I problemi al contorno per le equazioni differenziali di tipo elliptico, Ann. Scuola Norm. Sup. Pisa, # 3 (1958), V. 12, 247–358.

[42] A. Miranville and X. Wang, Attractors for nonautonomous nonhomogeneous Navier–Stokes equations, Nonlinearity 10 (1997), 1047–1061.

[43] S. M. Nikol'skii, Approximation of functions of several variables and imbedding theorems, Die Grundlehren der Mathematischen Wissenschaften, Band 205, Springer-Verlag, New York–Heidelberg, 1975.

[44] J. G. Oldroyd, Non-Newtonian flow of fluids and solids, in: Rheology: theory and applications, F. R. Eirich (Ed.), Academic Press Inc., Publishers, 1956.

[45] J. G. Oldroyd, On the formation of rheological equations of state, Proc. R. Soc. Lond, A200, 523–541, 1950.

[46] V. P. Orlov, P. E. Sobolevskii, On mathematical models of viscoelasticity with memory, Differential and Integral Equations, V.4, no. 1 (1991), 103–115.

[47] V. P. Orlov, P. E. Sobolevskii, Investigation of mathematical models of multidimensional viscoelastic media, Dokl. Akad. Nauk Ukrain. SSR Ser. A, no. 10 (1989), 31–35, 87 (Russian).

[48] V. P. Orlov, P. E. Sobolevskii, Investigation of mathematical models of multidimensional viscoelastic media, Preprint Akad. Nauk Ukrain. SSR, No. 27–29, Institute of applied problems of mechanics and mathematics, Lvov, 1989 (Russian).

[49] M. Reiner, Rheology, in: Handbuch der Physik, S. Flugge (Ed.), Bd. VI, Springer, 1958.

[50] M. Renardy, Existence of slow steady flows of viscoelastic fluids with differential constitutive equations, Z. angew. Math. Mech. V. 65 (1985), 449–451.

[51] M. Renardy, W. J. Hrusa, J. A. Nohel, Mathematical Problems in Viscoelasticity, Longman, London, 1987.

[52] G. de Rham, Variétés différentiables, Hermann, Paris, 1960.

[53] G. Sell, Global attractors for the three-dimensional Navier–Stokes equations, J. Dyn. Diff. Eq., V. 8 (1996), No. 1, 1–33.

[54] J. Simon, Compact sets in the space $L^p(0, T; B)$, Ann. Mat. Pura Appl., V. 146 (1987), 65–96.

[55] J. Simon, On the existence of the pressure for solutions of the variational Navier–Stokes equation, J. Mathematical Fluid Mech., 3 (1999), 225–234.

[56] J. Simon, Primitives de distributions et applications, Séminaire Analyse, Univ. Blaise Pascal, V. 6–7 (1991).

[57] G. F. Smith, On a Fundamental Error in Two Papers of C.-C. Wang "On Representations for Isotropic Functions, Parts I and II", Arch. Rational Mech. Anal., V. 36 (1970), 161–165.

[58] P. E. Sobolevskii, On the equations of parabolic type in a Banach space, Trudy Mos. Mat. Obsch., V.10 (1961), 297–350.

[59] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, New Jersey, 1970.

[60] R. Talhouk, Existence locale et unicité d'écoulements de fluides viscoélastiques dans des domaines non bornés, C.R. Acad. Sci. Paris. Serie I., t.328 (1999), 87–92.

[61] R. Temam, Navier–Stokes equations, Theory and numerical analysis, Studies in Mathematics and its Applications, Vol. 2, North-Holland Publishing Co., Amsterdam–New York–Oxford, 1977.

[62] R. Temam, Infinite-dimensional dynamical systems in mechanics and physics, 2nd edition, Springer-Verlag, 1997.

[63] C. Truesdell, A first course in rational continuum mechanics, The John Hopkins University, Baltimore, 1972.

[64] M. V. Turbin, Research of a mathematical model of low-concentrated aqueous polymer solutions, Abstr. Appl. Anal. 2006, Art. ID 12497, 27 pp.

[65] E. M. Turganbaev, Initial-boundary value problems for the equations of a viscoelastic fluid of Oldroyd type, Siberian Mathematical Journal, Vol. 36 (1995), No. 2, 389–403.

[66] M. I. Vishik, V. V. Chepyzhov, Trajectory and global attractors for 3D Navier–Stokes system, Mat. Zametki, V. 71 (2002), 194–213.

[67] D. A. Vorotnikov, On the motion of a nonlinear-viscous fluid in R^n, Vestnik VSU, Ser. phys.–math., no.1 (2002), p. 102–120 (in Russian).

[68] D. A. Vorotnikov, On the existence of weak stationary solutions of the boundary value problem in the Jeffreys model of motion of a viscoelastic medium, Izvestiya Vuzov, Ser. Math., no. 9 (2004), 13–17.

[69] D. A. Vorotnikov, Energy inequality and uniqueness of a weak solution of the initial-boundary value problem for the motion equations of viscoelastic medium, Vestnik VSU, Ser. phys.–math., no. 1 (2004), 96–102 (in Russian).

[70] D. A. Vorotnikov, Continuous dependence of solutions on data of initial-value problem for motion equations of nonlinear visoelastic medium, Vestnik VSU, Ser. phys.–math., no. 1 (2005), 148–151 (in Russian).

[71] D. A. Vorotnikov, Dissipative solutions for equations of viscoelastic diffusion in polymers, J. Math. Anal. Appl., Volume 339 (2008), 876–888.

[72] D. A. Vorotnikov, V. G. Zvyagin, On the existence of weak solutions for the initial-boundary value problem in the Jeffreys model of motion of a viscoelastic medium, Abstr. Appl. Anal., # 10 (2004), V. 2004, 815–829.

[73] D. A. Vorotnikov, V. G. Zvyagin, On the solvability of the initial-boundary value problem for the motion equations of nonlinear viscoelastic medium in the whole space, Nonlinear Anal. TMA, V. 58 (2004), 631–656.

[74] D. A. Vorotnikov, V. G. Zvyagin, On convergence of solutions of the regularized problem for motion equations of Jeffreys viscoelastic medium to solutions of the original problem, Journal of Mathematical Sciences, V.144 (2007), No.5, 4398–4408. Translated from Fundamentalnaya i prikladnaya matematika (FPM), V. 11 (2005), no. 4, 49–63.

[75] D. A. Vorotnikov, V. G. Zvyagin, On the trajectory and global attractors for the equations of motion of a visco-elastic medium, Russ. Math. Surv., V. 61 (2006), No. 2, 368–370.

[76] D. A. Vorotnikov, V. G. Zvyagin, Trajectory and global attractors of the boundary value problem for autonomous motion equations of viscoelastic medium, J. Math. Fluid Mech., V. 10 (2008), 19–44.

[77] D. A. Vorotnikov, V. G. Zvyagin, Uniform attractors for non-autonomous motion equations of viscoelastic medium, J. Math. Anal. Appl., Volume 325 (2007), Issue 1, 438–458.

[78] W. von Wahl, The equations of Navier–Stokes and abstract parabolic equations, Friedr. Vieweg und Sohn, Braunschweig, 1985.

[79] C.-C. Wang, A New Representation Theorem for Isotropic Functions: An Answer to Professor G. F. Smith's Criticism of my Papers on Representations for Isotropic Functions. Part 1. Scalar-valued Isotropic Functions, Arch. Rational Mech. Anal., V. 36 (1970), 166–197.

[80] C.-C. Wang, A New Representation Theorem for Isotropic Functions: An Answer to Professor G. F. Smith's Criticism of my Papers on Representations for Isotropic Functions. Part 2. Vector-valued Isotropic Functions, Symmetric Tensor-valued Isotropic Functions, and Skew-symmetric Tensor-valued Isotropic Functions, Arch. Rational Mech. Anal., V. 36 (1970), 198–223.

[81] C.-C. Wang, Corrigendum to My Recent Papers on "Representations for Isotropic Functions", Arch. Rational Mech. Anal., 1971, V. 43, 392–395.

[82] X. Wang. A remark on the characterization of the gradient of a distribution, Appl. Anal., V. 51 (1993), # 1–4, 35–40.

[83] W. L. Wilkinson, Non-Newtonian fluids, Pergamon Press, 1960.

[84] K. Yosida, Functional analysis, Sixth edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 123, Springer-Verlag, Berlin–New York, 1980.

[85] V. G. Zvyagin, On the solvability of some initial-boundary value problems for mathematical models of the motion of nonlinearly viscous and viscoelastic fluids, Sovrem. Mat. Fundam. Napravl., 2 (2003), 57–69 (electronic) (Russian); translation in J. Math. Sci. (N. Y.), 124 (2004), no. 5, 5321–5334.

[86] V. T. Dmitrienko, V. G. Zvyagin, Homotopy classification of a class of continuous mappings, Mat. Zametki 31 (1982), no. 5, 801–812, 816 (Russian).

[87] V. G. Zvyagin, V. T. Dmitrienko, Approximating–topological approach to investigation of problems of hydrodynamics. Navier–Stokes system, Editorial URSS, Moscow, 2004.

[88] V. G. Zvyagin, V. T. Dmitrienko, On the weak solutions of the regularized model of a viscoelastic fluid, Differencial'nye Uravnenija, V. 38 (2002), no. 12, 1633–1645.

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